

On Pointed Hopf Algebras with Weyl Groups of Exceptional Type

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Abstract

All -1 -type pointed Hopf algebras and central quantum linear spaces with Weyl groups of exceptional type are found. It is proved that every non -1 -type pointed Hopf algebra with real $G(H)$ is infinite dimensional and every central quantum linear space over finite group is finite dimensional. It is proved that except a few cases Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional.

2000 Mathematics Subject Classification: 16W30, 16G10

keywords: Quiver, Hopf algebra, Weyl group.

0 Introduction

This article is to contribute to the classification of finite-dimensional complex pointed Hopf algebras H with Weyl groups of exceptional type. The classification of finite dimensional pointed Hopf algebra with finite abelian groups has been completed (see [AS98, AS02, AS00, AS05, He06]). Papers [AG03, Gr00, AZ07, Fa07, AF06, AF07] considered some non-abelian cases, for example, symmetric group, dihedral group, alternating group and the Mathieu simple groups. It was shown in [HS] that every Nichols algebra of reducible Yetter-Drinfeld module over non-commutative finite simple group and symmetric group is infinite dimensional.

In this paper we find all -1 -type pointed Hopf algebras and quantum linear spaces with Weyl groups of exceptional type. We show that every non -1 -type pointed Hopf algebra is infinite dimensional and every quantum linear space is finite dimensional. It is desirable to do this in view of the importance of Weyl groups in the theories of Lie groups,

Lie algebras and algebraic groups. We first give the relation between the bi-one Nichols algebra $\mathfrak{B}(\mathcal{O}_s, \rho)$ introduced in [Gr00, AZ07, AHS08, AFZ] and the arrow Nichols algebra introduced in [CR97, CR02, ZYC, ZCZ]. [ZWCYa, ZWCYb] applied the software GAP to compute the representatives of conjugacy classes, centralizers of these representatives and character tables of these centralizers in Weyl groups of exceptional type. Using the results in [ZWCYa, ZWCYb] and the classification theorem of quiver Hopf algebras and Nichols algebras in [ZCZ, Theorem 1] we find all -1 -type pointed Hopf algebras and quantum linear spaces with Weyl groups of exceptional type. We prove that Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional except a few cases by applying [HS, Theorem 8.2, 8.6].

This paper is organized as follows. In section 1 it is shown that bi-one arrow Nichols algebras and $\mathfrak{B}(\mathcal{O}_s, \rho)$ introduced in [DPR, Gr00, AZ07, AHS08, AFZ] are the same up to isomorphisms. In section 2 it is proved that every non -1 -type pointed Hopf algebra with real $G(H)$ is infinite dimensional. In section 3 it is shown that every central quantum linear space is finite dimensional with an arrow PBW basis. In section 4 the programs to compute the representatives of conjugacy classes, centralizers of these representatives and character tables of these centralizers in Weyl groups of exceptional type are given. In section 5 all -1 -type bi-one Nichols algebras over Weyl groups of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are listed in tables. In section 6 all -1 -type bi-one Nichols algebras over Weyl groups of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are listed in tables. In section 7 all central quantum linear spaces over Weyl groups of exceptional type are found. In section 8 it is proved that except a few cases Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional.

Preliminaries And Conventions

Throughout this paper let k be the complex field; G be a finite group; \hat{G} denote the set of all isomorphic classes of irreducible representations of group G ; G^s denote the centralizer of s ; $Z(G)$ denote the center of G . For $h \in G$ and an isomorphism ϕ from G to G' , define a map ϕ_h from G to G' by sending x to $\phi(h^{-1}xh)$ for any $x \in G$. Let s^G or \mathcal{O}_s denote the conjugacy class containing s in G . The Weyl groups of E_6 , E_7 , E_8 , F_4 and G_2 are called *Weyl groups of exceptional type*. Let $\deg \rho$ denote the dimension of the representation space V for a representation (V, ρ) .

Let \mathbb{N} and \mathbb{Z} denote the sets of all positive integers and all integers, respectively. For a set X , we denote by $|X|$ the number of elements in X . If $X = \bigoplus_{i \in I} X_{(i)}$ as vector spaces, then we denote by ι_i the natural injection from $X_{(i)}$ to X and by π_i the corresponding projection from X to $X_{(i)}$. We will use μ to denote the multiplication of an algebra and

use Δ to denote the comultiplication of a coalgebra. For a (left or right) module and a (left or right) comodule, denote by α^- , α^+ , δ^- and δ^+ the left module, right module, left comodule and right comodule structure maps, respectively. The Sweedler's sigma notations for coalgebras and comodules are $\Delta(x) = \sum x_1 \otimes x_2$, $\delta^-(x) = \sum x_{(-1)} \otimes x_{(0)}$, $\delta^+(x) = \sum x_{(0)} \otimes x_{(1)}$.

A quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where Q_0 and Q_1 are the sets of vertices and arrows, respectively; s and t are two maps from Q_1 to Q_0 . For any arrow $a \in Q_1$, $s(a)$ and $t(a)$ are called *its start vertex and end vertex*, respectively, and a is called an *arrow* from $s(a)$ to $t(a)$. For any $n \geq 0$, an n -path or a path of length n in the quiver Q is an ordered sequence of arrows $p = a_n a_{n-1} \cdots a_1$ with $t(a_i) = s(a_{i+1})$ for all $1 \leq i \leq n-1$. Note that a 0-path is exactly a vertex and a 1-path is exactly an arrow. In this case, we define $s(p) = s(a_1)$, the start vertex of p , and $t(p) = t(a_n)$, the end vertex of p . For a 0-path x , we have $s(x) = t(x) = x$. Let Q_n be the set of n -paths. Let ${}^y Q_n^x$ denote the set of all n -paths from x to y , $x, y \in Q_0$. That is, ${}^y Q_n^x = \{p \in Q_n \mid s(p) = x, t(p) = y\}$.

A quiver Q is *finite* if Q_0 and Q_1 are finite sets. A quiver Q is *locally finite* if ${}^y Q_1^x$ is a finite set for any $x, y \in Q_0$.

Let $\mathcal{K}(G)$ denote the set of conjugate classes in G . A formal sum $r = \sum_{C \in \mathcal{K}(G)} r_C C$ of conjugate classes of G with cardinal number coefficients is called a *ramification* (or *ramification data*) of G , i.e. for any $C \in \mathcal{K}(G)$, r_C is a cardinal number. In particular, a formal sum $r = \sum_{C \in \mathcal{K}(G)} r_C C$ of conjugate classes of G with non-negative integer coefficients is a ramification of G .

For any ramification r and $C \in \mathcal{K}(G)$, since r_C is a cardinal number, we can choose a set $I_C(r)$ such that its cardinal number is r_C without loss of generality. Let $\mathcal{K}_r(G) := \{C \in \mathcal{K}(G) \mid r_C \neq 0\} = \{C \in \mathcal{K}(G) \mid I_C(r) \neq \emptyset\}$. If there exists a ramification r of G such that the cardinal number of ${}^y Q_1^x$ is equal to r_C for any $x, y \in G$ with $x^{-1}y \in C \in \mathcal{K}(G)$, then Q is called a *Hopf quiver with respect to the ramification data r* . In this case, there is a bijection from $I_C(r)$ to ${}^y Q_1^x$, and hence we write ${}^y Q_1^x = \{a_{y,x}^{(i)} \mid i \in I_C(r)\}$ for any $x, y \in G$ with $x^{-1}y \in C \in \mathcal{K}(G)$.

$(G, r, \vec{\rho}, u)$ is called a *ramification system with irreducible representations* (or RSR in short), if r is a ramification of G ; u is a map from $\mathcal{K}(G)$ to G with $u(C) \in C$ for any $C \in \mathcal{K}(G)$; $I_C(r, u)$ and $J_C(i)$ are sets with $|J_C(i)| = \deg(\rho_C^{(i)})$ and $I_C(r) = \{(i, j) \mid i \in I_C(r, u), j \in J_C(i)\}$ for any $C \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$; $\vec{\rho} = \{\rho_C^{(i)}\}_{i \in I_C(r, u), C \in \mathcal{K}_r(G)} \in \prod_{C \in \mathcal{K}_r(G)} (\widehat{G^{u(C)}})^{|I_C(r, u)|}$ with $\rho_C^{(i)} \in \widehat{G^{u(C)}}$ for any $i \in I_C(r, u)$, $C \in \mathcal{K}_r(G)$. In this paper we always assume that $I_C(r, u)$ is a finite set for any $C \in \mathcal{K}_r(G)$. Furthermore, if $\rho_C^{(i)}$ is a one dimensional representation for any $C \in \mathcal{K}_r(G)$, then $(G, r, \vec{\rho}, u)$ is called a *ramification system with characters* (or RSC $(G, r, \vec{\rho}, u)$ in short) (see [ZZC, Definition 1.8]). In this case, $a_{y,x}^{(i,j)}$ is written as $a_{y,x}^{(i)}$ in short since $J_C(i)$ has only one element.

For RSR $(G, r, \vec{\rho}, u)$, let $\chi_C^{(i)}$ denote the character of $\rho_C^{(i)}$ for any $i \in I_C(r, u)$, $C \in$

$\mathcal{K}_r(C)$. If ramification $r = r_C C$ and $I_C(r, u) = \{i\}$ then we say that $\text{RSR}(G, r, \vec{\rho}, u)$ is bi-one, written as $\text{RSR}(G, \mathcal{O}_s, \rho)$ with $s = u(C)$ and $\rho = \rho_C^{(i)}$ in short, since r only has one conjugacy class C and $|I_C(r, u)| = 1$. Quiver Hopf algebras, Nichols algebras and Yetter-Drinfeld modules, corresponding to a bi-one $\text{RSR}(G, r, \vec{\rho}, u)$, are said to be bi-one.

If $(G, r, \vec{\rho}, u)$ is an RSR, then it is clear that $\text{RSR}(G, \mathcal{O}_{u(C)}, \rho_C^{(i)})$ is bi-one for any $C \in \mathcal{K}$ and $i \in I_C(r, u)$, which is called a *bi-one* sub-RSR of $\text{RSR}(G, r, \vec{\rho}, u)$,

If $\phi : A \rightarrow A'$ is an algebra homomorphism and (M, α^-) is a left A' -module, then M becomes a left A -module with the A -action given by $a \cdot x = \phi(a) \cdot x$ for any $a \in A$, $x \in M$, called a *pullback* A -module through ϕ , written as ${}_\phi M$. Dually, if $\phi : C \rightarrow C'$ is a coalgebra homomorphism and (M, δ^-) is a left C -comodule, then M is a left C' -comodule with the C' -comodule structure given by $\delta'^- := (\phi \otimes \text{id})\delta^-$, called a *push-out* C' -comodule through ϕ , written as ${}^\phi M$.

If B is a Hopf algebra and M is a B -Hopf bimodule, then we say that (B, M) is a Hopf bimodule. For any two Hopf bimodules (B, M) and (B', M') , if ϕ is a Hopf algebra homomorphism from B to B' and ψ is simultaneously a B -bimodule homomorphism from M to ${}_\phi M'_\phi$ and a B' -bicomodule homomorphism from ${}^\phi M^\phi$ to M' , then (ϕ, ψ) is called a *pull-push* Hopf bimodule homomorphism. Similarly, we say that (B, M) and (B, X) are a Yetter-Drinfeld (YD) module and YD Hopf algebra, respectively, if M is a YD B -module and X is a braided Hopf algebra in YD category ${}^B_B \mathcal{YD}$. For any two YD modules (B, M) and (B', M') , if ϕ is a Hopf algebra homomorphism from B to B' , and ψ is simultaneously a left B -module homomorphism from M to ${}_\phi M'$ and a left B' -comodule homomorphism from ${}^\phi M$ to M' , then (ϕ, ψ) is called a pull-push YD module homomorphism. For any two YD Hopf algebras (B, X) and (B', X') , if ϕ is a Hopf algebra homomorphism from B to B' , ψ is simultaneously a left B -module homomorphism from X to ${}_\phi X'$ and a left B' -comodule homomorphism from ${}^\phi X$ to X' , meantime, ψ also is algebra and coalgebra homomorphism from X to X' , then (ϕ, ψ) is called a pull-push YD Hopf algebra homomorphism (see [ZZC, the remark after Th.4]).

For $s \in G$ and $(\rho, V) \in \widehat{G^s}$, here is a precise description of the YD module $M(\mathcal{O}_s, \rho)$, introduced in [Gr00, AZ07]. Let $t_1 = s, \dots, t_m$ be a numeration of \mathcal{O}_s , which is a conjugacy class containing s , and let $g_i \in G$ such that $g_i \triangleright s := g_i s g_i^{-1} = t_i$ for all $1 \leq i \leq m$. Then $M(\mathcal{O}_s, \rho) = \bigoplus_{1 \leq i \leq m} g_i \otimes V$. Let $g_i v := g_i \otimes v \in M(\mathcal{O}_s, \rho)$, $1 \leq i \leq m$, $v \in V$. If $v \in V$ and $1 \leq i \leq m$, then the action of $h \in G$ and the coaction are given by

$$\delta(g_i v) = t_i \otimes g_i v, \quad h \cdot (g_i v) = g_j (\gamma \cdot v), \quad (0.1)$$

where $h g_i = g_j \gamma$, for some $1 \leq j \leq m$ and $\gamma \in G^s$. The explicit formula for the braiding is then given by

$$c(g_i v \otimes g_j w) = t_i \cdot (g_j w) \otimes g_i v = g_{j'} (\gamma \cdot v) \otimes g_i v \quad (0.2)$$

for any $1 \leq i, j \leq m$, $v, w \in V$, where $t_i g_j = g_{j'} \gamma$ for unique j' , $1 \leq j' \leq m$ and $\gamma \in G^s$. Let $\mathfrak{B}(\mathcal{O}_s, \rho)$ denote $\mathfrak{B}(M(\mathcal{O}_s, \rho))$. $M(\mathcal{O}_s, \rho)$ is a simple YD module (see [AZ07, Section 1.2]). Furthermore, if χ is the character of ρ , then we also denote $\mathfrak{B}(\mathcal{O}_s, \rho)$ by $\mathfrak{B}(\mathcal{O}_s, \chi)$.

1 Relation between bi-one arrow Nichols algebras and $\mathfrak{B}(\mathcal{O}_s, \rho)$

In this section it is shown that bi-one arrow Nichols algebras and $\mathfrak{B}(\mathcal{O}_s, \rho)$ introduced in [Gr00, AZ07, AHS08, AFZ] are the same up to isomorphisms.

For any $\text{RSR}(G, r, \vec{\rho}, u)$, we can construct an arrow Nichols algebra $\mathfrak{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ (see [ZCZ, Pro. 2.4]), written as $\mathfrak{B}(G, r, \vec{\rho}, u)$ in short. Let us recall the precise description of arrow YD module. For an $\text{RSR}(G, r, \vec{\rho}, u)$ and a kG -Hopf bimodule $(kQ_1^c, G, r, \vec{\rho}, u)$ with the module operations α^- and α^+ , define a new left kG -action on kQ_1 by

$$g \triangleright x := g \cdot x \cdot g^{-1}, \quad g \in G, x \in kQ_1,$$

where $g \cdot x = \alpha^-(g \otimes x)$ and $x \cdot g = \alpha^+(x \otimes g)$ for any $g \in G$ and $x \in kQ_1$. With this left kG -action and the original left (arrow) kG -coaction δ^- , kQ_1 is a Yetter-Drinfeld kG -module. Let $Q_1^1 := \{a \in Q_1 \mid s(a) = 1\}$, the set of all arrows with starting vertex 1. It is clear that kQ_1^1 is a Yetter-Drinfeld kG -submodule of kQ_1 , denoted by $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$, called the arrow YD module.

Lemma 1.1. *For any $s \in G$ and $\rho \in \widehat{G^s}$, there exists a bi-one arrow Nichols algebra $\mathfrak{B}(G, r, \vec{\rho}, u)$ such that*

$$\mathfrak{B}(\mathcal{O}_s, \rho) \cong \mathfrak{B}(G, r, \vec{\rho}, u)$$

as graded braided Hopf algebras in ${}^{kG}_k \mathcal{YD}$.

Proof. Assume that V is the representation space of ρ with $\rho(g)(v) = g \cdot v$ for any $g \in G, v \in V$. Let $C = \mathcal{O}_s$, $r = r_C C$, $r_C = \deg \rho$, $u(C) = s$, $I_C(r, u) = \{1\}$ and $(v)\rho_C^{(1)}(h) = \rho(h^{-1})(v)$ for any $h \in G, v \in V$. We get a bi-one arrow Nichols algebra $\mathfrak{B}(G, r, \vec{\rho}, u)$.

We now only need to show that $M(\mathcal{O}_s, \rho) \cong (kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ in ${}^{kG}_k \mathcal{YD}$. We recall the notation in [ZCZ, Proposition 1.2]. Assume $J_C(1) = \{1, 2, \dots, n\}$ and $X_C^{(1)} = V$ with basis $\{x_C^{(1,j)} \mid j = 1, 2, \dots, n\}$ without loss of generality. Let v_j denote $x_C^{(1,j)}$ for convenience. In fact, the left and right coset decompositions of G^s in G are

$$G = \bigcup_{i=1}^m g_i G^s \quad \text{and} \quad G = \bigcup_{i=1}^m G^s g_i^{-1}, \quad (1.1)$$

respectively.

Let ψ be a map from $M(\mathcal{O}_s, \rho)$ to $(kQ_1^{\text{ad}}, \text{ad}(G, r, \vec{\rho}, u))$ by sending $g_i v_j$ to $a_{t_i, 1}^{(1, j)}$ for any $1 \leq i \leq m, 1 \leq j \leq n$. Since the dimension is mn , ψ is a bijective. See

$$\begin{aligned} \delta^-(\psi(g_i v_j)) &= \delta^-(a_{t_i, 1}^{(1, j)}) \\ &= t_i \otimes a_{t_i, 1}^{(1, j)} = (id \otimes \psi) \delta^-(g_i v_j). \end{aligned}$$

Thus ψ is a kG -comodule homomorphism. For any $h \in G$, assume $hg_i = g_{i'} \gamma$ with $\gamma \in G^s$. Thus $g_i^{-1} h^{-1} = \gamma^{-1} g_{i'}^{-1}$, i.e. $\zeta_i(h^{-1}) = \gamma^{-1}$, where ζ_i was defined in [ZZC, (0.3)]. Since $\gamma \cdot x^{(1, j)} \in V$, there exist $k_{C, h^{-1}}^{(1, j, p)} \in k$, $1 \leq p \leq n$, such that $\gamma \cdot x^{(1, j)} = \sum_{p=1}^n k_{C, h^{-1}}^{(1, j, p)} x^{(1, p)}$. Therefore

$$\begin{aligned} x^{(1, j)} \cdot \zeta_i(h^{-1}) &= \gamma \cdot x^{(1, j)} \quad (\text{by definition of } \rho_C^{(1)}) \\ &= \sum_{p=1}^n k_{C, h^{-1}}^{(1, j, p)} x^{(1, p)}. \end{aligned} \tag{1.2}$$

See

$$\begin{aligned} \psi(h \cdot g_i v_j) &= \psi(g_{i'}(\gamma v_j)) \\ &= \psi(g_{i'}(\sum_{p=1}^n k_{C, h^{-1}}^{(1, j, p)} v_p)) \\ &= \sum_{p=1}^n k_{C, h^{-1}}^{(1, j, p)} a_{t_{i'}, 1}^{(1, p)} \end{aligned}$$

and

$$\begin{aligned} h \triangleright (\psi(g_i v_j)) &= h \triangleright (a_{t_i, 1}^{(1, j)}) \\ &= a_{ht_i, h}^{(1, j)} \cdot h^{-1} \\ &= \sum_{p=1}^n k_{C, h^{-1}}^{(1, j, p)} a_{t_{i'}, 1}^{(1, p)} \quad (\text{by [ZCZ, Pro.1.2] and (1.2)}). \end{aligned}$$

Therefore ψ is a kG -module homomorphism. \square

Therefore we also say that $\mathfrak{B}(\mathcal{O}_s, \rho)$ is a bi-one Nichols Hopf algebra.

Remark 1.2. The representation ρ in $\mathfrak{B}(\mathcal{O}_s, \rho)$ introduced in [Gr00, AZ07] and $\rho_C^{(i)}$ in RSR are different. $\rho(g)$ acts on its representation space from the left and $\rho_C^{(i)}(g)$ acts on its representation space from the right.

$s \in G$ is *real* if s and s^{-1} are in the same conjugacy class. If every element in G is real, then G is real. Obviously, Weyl groups are real.

Lemma 1.3. Assume that $s \in G$ is real and χ is the character of $\rho \in \widehat{G^s}$. If $\chi(s) \neq -\deg(\rho)$ or the order of s is odd, then $\dim \mathfrak{B}(\mathcal{O}_s, \rho) = \infty$.

Proof. If the order of s is odd, it follows from [AZ07, Lemma 2.2] and [AF07, Lemma 1.3]. Now assume that $\chi(s) \neq -\deg(\rho)$. Since $\rho(s) = q_{ss}\text{id}$, $\chi(s) = q_{ss}(\deg(\rho))$. Therefore $q_{ss} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_s, \rho) = \infty$ by [AZ07, Lemma 2.2] and [AF07, Lemma 1.3]. \square .

Lemma 1.4. $(kG, \mathfrak{B}(\mathcal{O}_s, \rho)) \cong (kG', \mathfrak{B}(\mathcal{O}_{s'}, \rho'))$ as graded pull-push YD Hopf algebras if and only if there exist $h \in G$ and a group isomorphism ϕ from G to G' such that $\phi(h^{-1}sh) = s'$ and $\rho'\phi_h \cong \rho$, where $\phi_h(g) = \phi(h^{-1}gh)$ for any $g \in G$.

Proof. Let C and C' be conjugacy classes of G and G' , respectively; $r = r_C C$ and $r' = r_{C'} C'$ be ramifications of G and G' , respectively. Applying Lemma 1.1, we only need show that $(kG, \mathfrak{B}(G, r, \vec{\rho}, u)) \cong (kG', \mathfrak{B}(G', r', \vec{\rho}', u'))$ as graded pull-push YD Hopf algebras if and only if there exist $h \in G$ and a group automorphism group isomorphism ϕ from G to G' such that $\phi(h^{-1}u(C)h) = u'(C')$ and $\rho'_{C'}^{(i')} \phi_h \cong \rho_C^{(i)}$. Applying [ZCZ, Theorem 4], we only need show that $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$ if and only if there exist $h \in G$ and a group isomorphism ϕ from G to G' such that $\phi(h^{-1}u(C)h) = u'(C')$ and $\rho'_{C'}^{(i')} \phi_h \cong \rho_C^{(i)}$. This is clear. \square

If we define a relation on group G as follows: $x \sim y$ if and only if there exists a group automorphism ϕ of G such that $\phi(x)$ and y are contained in the same conjugacy class, then this is an equivalent relation. Let set $\{s_i \mid i \in \Omega\}$ denote all representatives of the equivalent classes, which is called the *representative system of conjugacy classes* of G under isomorphism relations, or the representative system of iso-conjugacy classes of G in short.

Proposition 1.5. Let $\{s_i \mid i \in \Omega\} \subseteq G$ be the representative system of iso-conjugacy classes of G . Then $\{\mathfrak{B}(\mathcal{O}_{s_i}, \rho) \mid i \in \Omega, \rho \in \widehat{G^{s_i}}\}$ are all representatives of the bi-one Nichols algebra over G , up to graded pull-push YD Hopf algebra isomorphisms.

Proof. If $\mathfrak{B}(\mathcal{O}_s, \rho)$ is a bi-one Nichols Hopf algebra over G , then there exist $i \in \Omega$, $\phi \in \text{Aut}(G)$ and $h \in G$ such that $\phi_h(s) = s_i$. Let $\rho' = \rho(\phi_h)^{-1}$. By Lemma 1.4, $(kG, \mathfrak{B}(\mathcal{O}_s, \rho)) \cong (kG, \mathfrak{B}(\mathcal{O}_{s_i}, \rho'))$ as graded pull-push YD Hopf algebras.

It follows from Lemma 1.4 that $(kG, \mathfrak{B}(\mathcal{O}_{s_i}, \rho))$ and $(kG, \mathfrak{B}(\mathcal{O}_{s_j}, \rho'))$ are not graded pull-push YD Hopf algebra isomorphisms when $i \neq j$ and $i, j \in \Omega$. \square

2 Diagram

In this section it is proved that every non -1 -type pointed Hopf algebra with real $G(H)$ is infinite dimensional.

If H is a graded Hopf algebra, then there exists the diagram of H , written $\text{diag}(H)$, (see [ZZC, Section 3.1] and [Ra]). If H is a pointed Hopf algebra, then the coradical filtration Hopf algebra $\text{gr}(H)$ is a graded Hopf algebra. So $\text{gr}(H)$ has the diagram, written

$\text{diag}_{\text{filt}}(H)$, called the *filter diagram* of H . $\text{diag}_{\text{filt}}(H)$ is written as $\text{diag}(H)$ in short when it does not cause confusion (see [AS98, Introduction]).

A graded coalgebra $C = \bigoplus_{n=0}^{\infty} C_{(n)}$ is strictly graded if $C_{(0)} = k$ and $C_{(1)} = P(C)$ (see [Sw, P232]).

Proposition 2.1. *If $H = \bigoplus_{n=0}^{\infty} H_{(n)}$ is a graded Hopf algebra and $R := \text{diag}(H)$ is strictly graded as coalgebras, then $H \cong \text{gr}H$ as graded Hopf algebras.*

Proof. By [AS98, Lemma 2.5], H is coradically graded, i.e. $H_m = \bigoplus_{i=0}^m H_{(i)}$ for $m = 0, 1, 2, \dots$, where $H_0 \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$ is the coradical filtration of H . Define a map ψ from H to $\text{gr}H$ by sending a to $a + H_{m-1}$ for any $a \in H_{(m)}$ and $m = 0, 1, 2, 3, \dots$. Note $H_{-1} := 0$. Obviously, ψ is bijective. If $a \in H_{(m)}$, then there exist $a_s^{(j)}, b_s^{(j)} \in H_{(j)}$ for $0 \leq j \leq m, 1 \leq s \leq n_j$, such that $\Delta(a) = \sum_{i=0}^m \sum_{s=1}^{n_i} a_s^{(i)} \otimes b_s^{(m-i)}$. See

$$\begin{aligned} (\psi \otimes \psi)\Delta(a) &= (\psi \otimes \psi) \sum_{i=0}^m \sum_{s=1}^{n_i} a_s^{(i)} \otimes b_s^{(m-i)} \\ &= \sum_{i=0}^m \sum_{s=1}^{n_i} (a_s^{(i)} + H_{i-1}) \otimes (b_s^{(m-i)} + H_{m-i-1}) \\ &= \Delta(a + H_{m-1}) \\ &\quad \text{(by the definition of comultiplication of } \text{gr}H \text{ in [Sw, P229])} \\ &= \Delta\psi(a). \end{aligned}$$

Thus ψ is a coalgebra homomorphism. Similarly, ψ is an algebra homomorphism. \square

Consequently, every pointed Hopf algebra of type one (since its diagram is Nichols algebra, see [ZCZ, Section 2]) is isomorphic to its filtration Hopf algebra as graded Hopf algebras.

Lemma 2.2. *If R is a graded braided Hopf algebra in ${}^{kG}_G \mathcal{YD}$ and is strictly graded as coalgebra gradations, then the subalgebra \bar{R} generated by $R_{(1)}$ as algebras is a Nichols algebra. Furthermore, \bar{R} generated by $R_{(1)}$ as algebras in R is a Nichols algebra when R is the filter diagram of a pointed Hopf algebra H .*

Proof. We show the first claim by the following steps. Let

$$x = x^{(1)}x^{(2)} \cdots x^{(n)} \quad \text{and} \quad y = x^{(1)}, \quad z = x^{(2)} \cdots x^{(n)}$$

with $x \in R, x^{(i)} \in R_{(1)}$ for $i = 1, 2, \dots, n$.

(i) \bar{R} is kG -submodule of R . In fact $h \cdot x = h \cdot x^{(1)}x^{(2)} \cdots x^{(n)} = (h \cdot x^{(1)})(h \cdot x^{(2)}) \cdots (h \cdot x^{(n)}) \in R_{(1)}R_{(1)} \cdots R_{(1)} \subseteq \bar{R}$ for any $h \in G$.

(ii) \bar{R} is kG -subcomodule of R . We use induction on n to show $\delta^-(x) \in kG \otimes \bar{R}$. When $n = 1$, it is clear. Assume $n > 1$. $\delta^-(x) = \delta^-(yz) = \sum y_{(-1)}z_{(-1)} \otimes y_{(0)}z_{(0)} \in kG \otimes \bar{R}$.

(iii) \bar{R} is a subcoalgebra of R . We use induction on n to show $\Delta(x) \in \bar{R} \otimes \bar{R}$. When $n = 1$ it is clear. Assume $n > 1$.

$$\begin{aligned}\Delta_R(x) &= \Delta_R(yz) \\ &= \sum_{(z)} yz_1 \otimes z_2 + \sum_{(z),(y)} y_{(-1)} \cdot z_1 \otimes y_{(0)}z_2,\end{aligned}$$

which implies $\Delta_R(x) \in \bar{R} \otimes \bar{R}$.

For the second claim, since R is strictly graded as coalgebra gradations (see [AS98, Lemma 2.3 and Lemma 2.4]), \bar{R} is a Nichols algebra by the first claim. \square

Remark 2.3. By [AS02, Cor.2.3] $\bar{R} \cong \mathfrak{B}(\text{diag}_{\text{filt}}(H)_{(1)})$ as graded braided Hopf algebra in ${}^k_G\mathcal{YD}$, where \bar{R} is the same as in Lemma 2.2. There exists an $\text{RSR}(G, r, \vec{\rho}, u)$ such that $\mathfrak{B}(G, r, \vec{\rho}, u) \cong \mathfrak{B}(\text{diag}_{\text{filt}}(H)_{(1)})$ as graded braided Hopf algebra in ${}^k_G\mathcal{YD}$, by [ZCZ, Pro. 2.4]. We call $\mathfrak{B}(\text{diag}_{\text{filt}}(H)_{(1)})$ and $\text{RSR}(G, r, \vec{\rho}, u)$ the Nichols algebra and RSR of H , respectively.

Definition 2.4. (i) $\text{RSR}(G, r, \vec{\rho}, u)$ is of -1 -type, if $u(C)$ is real and the order of $u(C)$ is even with $\chi_C^{(i)}(u(C)) = -\chi_C^{(i)}(1)$ (i.e. $\chi_C^{(i)}(u(C)) = -\deg \rho_C^{(i)}$) for any $C \in \mathcal{K}_r(G)$ and any $i \in I_C(r, u)$.

(ii) Nichols algebra R over group G is of -1 -type if there exists -1 -type $\text{RSR}(G, r, \vec{\rho}, u)$ such that $R \cong \mathfrak{B}(G, r, \vec{\rho}, u)$ as graded pull-push YD Hopf algebras.

(iii) Pointed Hopf algebra H with group $G = G(H)$ is of -1 -type if the Nichols algebra of H is of -1 -type.

Proposition 2.5. (i) If $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$ and $\text{RSR}(G, r, \vec{\rho}, u)$ is of -1 -type, then so is $\text{RSR}(G', r', \vec{\rho}', u')$.

(ii) If $(kG, R) \cong (kG', R')$ as graded pull-push YD Hopf algebras and R is of -1 -type, then so is R' , where R and R' are Nichols algebras over group algebras kG and kG' , respectively.

(iii) If pointed Hopf algebras H and H' are isomorphic as Hopf algebras and H is of -1 -type, then so is H' .

Proof. (i) There exist a group isomorphism $\phi : G \rightarrow G'$, an element $h_C \in G$ such that $\phi(h_C^{-1}u(C)h_C) = u'(\phi(C))$ for any $C \in \mathcal{K}(G)$ and a bijective map $\phi_C : I_C(r, u) \rightarrow I_{\phi(C)}(r', u')$ such that $\rho_C^{(i)} \cong \rho'_{\phi(C)}(\phi_C^{(i)})\phi_{h_C}$ for any $i \in I_C(r, u)$. Therefore

$$\begin{aligned}\chi'_{\phi(C)}(\phi_C^{(i)})(u'(\phi(C))) &= \chi'_{\phi(C)}(\phi_C^{(i)})(\phi(h_C^{-1}u(C)h_C)) \\ &= \chi_C^{(i)}(u(C)) \quad (\text{by the isomorphism}) \\ &= -\chi_C^{(i)}(1) \quad (\text{by the definition of } -1\text{-type}) \\ &= -\chi'_{\phi(C)}(\phi_C^{(i)})(\phi_{h_C}(1)) = -\chi'_{\phi(C)}(\phi_C^{(i)})(1),\end{aligned}$$

which proves the claim.

(ii) By [ZCZ, Pro.2.4], there exist two $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G', r', \vec{\rho}', u')$ such that $R \cong \mathfrak{B}(G, r, \vec{\rho}, u)$ and $R' \cong \mathfrak{B}(G', r', \vec{\rho}', u')$ as graded YD Hopf algebras. Thus $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$ by [ZCZ, Theorem 4]. It follows from Definition 2.4 and Part (i) that $\text{RSR}(G', r', \vec{\rho}', u')$ is of -1 -type.

(iii) It is clear that $\text{gr}H \cong \text{gr}H'$ as graded Hopf algebras. Thus $(kG, R) \cong (kG', R')$ as graded pull-push YD Hopf algebras by [ZCZ, Lemma 3.1], where kG and kG' are the coradicals of H and H' , respectively; $R = \text{diag}H$ and $R' = \text{diag}H'$. Let \bar{R} and \bar{R}' denote the subalgebras generated by $R_{(1)}$ and $R'_{(1)}$ as algebras in R and R' , respectively. It is clear that $(kG, \bar{R}) \cong (kG', \bar{R}')$ as graded pull-push YD Hopf algebras. It follows from Part (ii) that H' is of -1 -type. \square

In fact, the proof of Part (iii) above shows that if two pointed Hopf algebras are isomorphic, then their Nichols algebras are graded pull-push isomorphic. Similarly, we can prove that their RSR' s are isomorphic.

Proposition 2.6. *If H is a pointed Hopf algebra with real $G = G(H)$ and is not of -1 -type, then H is infinite dimensional.*

Proof. Let R be the (filter) diagram of H . By Lemma 2.2, \bar{R} generated by $R_{(1)}$ as algebras in R is a Nichols algebra. By [ZCZ, Pro.2.4 (ii)], there exists an $\text{RSR}(G, r, \vec{\rho}, u)$ such that $\bar{R} \cong \mathfrak{B}(G, r, \vec{\rho}, u)$ is graded pull-push YD Hopf algebra isomorphism. By assumption, there exist $C \in \mathcal{K}_r(G)$ and $i \in I_C(r, u)$ such that $\chi_C^{(i)}(u(C)) \neq -\deg(\rho_C^{(i)})$ or the order of $u(C)$ is odd. It follows from Lemma 1.3 that the bi-one Nichols algebra $\mathfrak{B}(G, r', \vec{\rho}', u')$ is infinite dimensional, where ramification $r' = r'_C C$, $\rho'_C^{(i)} = \rho_C^{(i)}$, $u'(C) = u(C)$, $I_C(r', u') \subseteq I_C(r, u)$ with $|I_C(r', u')| = 1$. Let Q' be a sub-quiver of Q with $Q'_0 = Q_0$ and $Q'_1 := \{a_{y,x}^{(i,j)} \mid x^{-1}y \in C, j \in J_C(i)\}$. Since $(k(Q')_1^1, \text{ad}(G, r', \vec{\rho}', u'))$ is a braided subspace of $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$, we have $\dim \mathfrak{B}(G, r, \vec{\rho}, u) = \infty$ and H is infinite dimensional. \square

$\text{RSR}(G, r, \vec{\rho}, u)$ is said to be of *infinite type* if $\mathfrak{B}(G, r, \vec{\rho}, u)$ is infinite dimensional. Otherwise, it is said to be of *finite type*. For any $\text{RSR}(G, r, \vec{\rho}, u)$, according to the proof above, if there exist $C \in \mathcal{K}_r(G)$ and $i \in I_C(r, u)$ such that $\dim \mathfrak{B}(\mathcal{O}_{u(C)}, \rho_C^{(i)}) = \infty$, then $\dim \mathfrak{B}(G, r, \vec{\rho}, u) = \infty$. In this case $\text{RSR}(G, r, \vec{\rho}, u)$ is said to be of *essentially infinite type*. Otherwise, it is said to be of *non-essentially infinite type*. For example, non -1 -type RSR over real group is of essentially infinite type. However, it is an open problem whether $\text{RSR}(G, r, \vec{\rho}, u)$ is of finite type when it is of non-essentially infinite type, although paper [AHS08] gave a partial solution to this problem.

3 Generalized quantum linear spaces

In this section it is shown that every central quantum linear space is finite dimensional with an arrow PBW basis.

Let σ denote the braiding of the braided tensor category (\mathcal{C}, σ) . If A and B are two objects of \mathcal{C} and $\sigma_{A,B}\sigma_{B,A} = \text{id}_{B \otimes A}$ and $\sigma_{B,A}\sigma_{A,B} = \text{id}_{A \otimes B}$ then σ is said to be *symmetric* on pair (A, B) . Furthermore, if $A = B$, then σ is said to be symmetric on object A , in short, or A is said to be quantum symmetric.

Every arrow YD module $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ has a decomposition of simple YD modules:

$$(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u)) = \bigoplus_{C \in \mathcal{K}_r(G), i \in I_C(r, u)} kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}) \quad (3.1)$$

and $\sigma_{C^{(i)}, D^{(j)}}$ is a map from $kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}) \otimes kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)})$ to $kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)}) \otimes kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)})$, where $kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}) := k\{a_{x,1}^{(i,j)} \mid x \in \mathcal{O}_{u(C)}, i \in I_C(r, u), j \in J_C(i)\}$ and $\sigma_{C^{(i)}, D^{(j)}}$ denotes $\sigma_{kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}), kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)})}$ for any $i \in I_C(r, u)$, $j \in I_D(r, u)$.

Every YD module over kG has a decomposition (3.1) since every YD module is isomorphic to an arrow YD module by [ZCZ, Pro. 2.4], which shows every YD module over kG is completely reducible (see [AZ07, Section 1.2]).

Definition 3.1. An $\text{RSR}(G, r, \vec{\rho}, u)$ is said to be *quantum symmetric* if $\sigma_{C^{(i)}, D^{(j)}} = \sigma_{D^{(j)}, C^{(i)}}^{-1}$, i.e. σ is symmetric on pair $(kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}), kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)}))$, for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$ and $j \in I_D(r, u)$.

An $\text{RSR}(G, r, \vec{\rho}, u)$ is said to be *quantum weakly symmetric* if $\sigma_{C^{(i)}, D^{(j)}} = (\sigma_{D^{(j)}, C^{(i)}})^{-1}$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$ and $j \in I_D(r, u)$ with $(C, i) \neq (D, j)$ (i.e. either $C \neq D$ or $i \neq j$).

Proposition 3.2. If a non-essentially infinite $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum weakly symmetric, then $\text{RSR}(G, r, \vec{\rho}, u)$ is a finite type.

Proof. It follows from [Gr00, Theorem 2.2]. \square

Lemma 3.3. (i) Assume that H is a Hopf algebra with an invertible antipode and M is a YD H -module, Then the braiding σ of ${}^H_H\mathcal{YD}$ is symmetric on M if and only if σ is symmetric on $\mathfrak{B}(M)$.

(ii) The following conditions are equivalent.

(1) $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum symmetric.

(2) The braiding σ of ${}^{kG}_{kG}\mathcal{YD}$ on the arrow YD module $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ is symmetric.

(3) The braiding σ is symmetric on $\mathfrak{B}(G, r, \vec{\rho}, u)$.

(4) $\sigma^2(a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')}) = a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')}$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in J_C(i)$, $j' \in J_D(i')$.

(5) $a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')} = (xyx^{-1} \triangleright a_{x,1}^{(i,j)}) \otimes (x \triangleright a_{y,1}^{(i',j')})$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in J_C(i)$, $j' \in J_D(i')$.

(iii) The following conditions are equivalent.

(1) $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum weakly symmetric.

(2) $\sigma^2(a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')}) = a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')}$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in J_C(i)$, $j' \in J_D(i')$ with $(C, i) \neq (D, i')$.

(3) $a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')} = (xyx^{-1} \triangleright a_{x,1}^{(i,j)}) \otimes (x \triangleright a_{y,1}^{(i',j')})$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in J_C(i)$, $j' \in J_D(i')$ with $(C, i) \neq (D, i')$.

Proof. (i) It is clear since M generates $\mathfrak{B}(M)$ as algebras.

(ii) It follows from Definition 3.1 that (1) and (2) are equivalent. Part (i) implies that (2) and (3) are equivalent. Obviously, (4) and (2) are equivalent. Since $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ is a YD module, we have

$$\sigma^2(a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')}) = (xyx^{-1} \triangleright a_{x,1}^{(i,j)}) \otimes (x \triangleright a_{y,1}^{(i',j')}). \quad (3.2)$$

Therefore, (4) and (5) are equivalent.

(iii) It follows from (3.2) that (2) and (3) are equivalent. Obviously (1) and (2) according to the definition. \square

Lemma 3.4. For $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in J_C(i)$, $j' \in J_D(i')$, assume that $\rho_C^{(i)}$ and $\rho_D^{(i')}$ are one dimensional representations; The coset decomposition of $G^{u(C)}$ and $G^{u(D)}$ in G are

$$G = \bigcup_{\theta \in \Theta_C} G^{u(C)} g_\theta, \text{ and } G = \bigcup_{\eta \in \Theta_D} G^{u(D)} h_\eta,$$

respectively; $x = g_\theta^{-1} u(C) g_\theta$ and $y = h_\eta^{-1} u(D) h_\eta$; $g_\theta y^{-1} = \zeta_\theta(y^{-1}) g_\theta$ and $h_\eta x^{-1} = \zeta_\eta(x^{-1}) h_\eta$ with $\zeta_\theta(y^{-1}) \in G^{u(C)}$ and $\zeta_\eta(x^{-1}) \in G^{u(D)}$.

Then

$$a_{x,1}^{(i,j)} \otimes a_{y,1}^{(i',j')} = (xyx^{-1} \triangleright a_{x,1}^{(i,j)}) \otimes (x \triangleright a_{y,1}^{(i',j')}) \quad (3.3)$$

if and only if

$$xy = yx \quad \text{and} \quad \rho_C^{(i)}(\zeta_\theta(y^{-1})) \rho_D^{(i')}(\zeta_\eta(x^{-1})) = 1 \quad (3.4)$$

Proof. By [ZCZ, Pro. 1.2] or [ZZC, Pro. 1.9], $(xyx^{-1} \triangleright a_{x,1}^{(i,j)}) \otimes (x \triangleright a_{y,1}^{(i',j')}) = \alpha a_{xyxy^{-1}x^{-1},1}^{(i,j)} \otimes a_{xyx^{-1},1}^{(i',j')}$, where $\alpha \in k$. Thus (3.3) holds if and only if $xy = yx$ and $\alpha = 1$. By [ZCZ, Pro. 1.2], $\alpha = 1$ if and only if (3.4) holds. \square

Proposition 3.5. *If $\text{RSC}(G, r, \overrightarrow{\chi}, u)$ is non-essentially infinite and (3.4) holds for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$ with $(C, i) \neq (D, i')$, then $\text{RSR}(G, r, \overrightarrow{\rho}, u)$ is quantum weakly symmetric. Therefore $\text{RSR}(G, r, \overrightarrow{\rho}, u)$ is a finite type.*

Proof. It follows from Lemma 3.4, Lemma 3.3 and Proposition 3.2. \square

If $0 \neq q \in k$ and $0 \leq i \leq n < \text{ord}(q)$ (the order of q), we set $(0)_q! = 1$,

$$\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}, \quad \text{where } (n)_q! = \prod_{1 \leq i \leq n} (i)_q, \quad (n)_q = \frac{q^n - 1}{q - 1}.$$

In particular, $(n)_q = n$ when $q = 1$.

Lemma 3.6. *In $kQ^c(G, r, \overrightarrow{\rho}, u)$, we have the following results.*

- (i) *If $C = \{g\} \in \mathcal{K}_r(G)$ with $i \in I_C(r, u)$, then there exists $0 \neq q \in k$ such that $\rho_C^{(i)}(g) = q \text{ id}$ and $a_{y,x}^{(i,j)} \cdot h = qa_{yh,xh}^{(i,j)}$ for any $x^{-1}y \in C$, $h \in G$, $j \in J_C(i)$.*
- (ii) *If $a_{w_0,v_0}^{(i,j)} \cdot h = qa_{w_0h,v_0h}^{(i,j)}$ for some $v_0, w_0 \in G$, $h \in G^{u(C)}$, $q \in k$ with $v_0^{-1}w_0 \in C \in \mathcal{K}_r(G)$, then $a_{w,v}^{(i,j)} \cdot h = qa_{w_0h,v_0h}^{(i,j)}$ for any $v, w \in G$ with $v^{-1}w \in C$*

Proof. (i) It follows from [ZCZ, Pro. 1.2].

(ii) Let $X_C^{(i)}$ be a representation space of $\rho_C^{(i)}$ and $\{x_C^{(i,j)} \mid j \in J_C(i)\}$ a k -basis of $X_C^{(i)}$. By the proof of [ZCZ, Pro. 1.2], $a_{w,v}^{(i,j)} \cdot h = qa_{wh,vh}^{(i,j)}$ since $x_C^{(i,j)} \cdot \zeta_\theta(h) = qx_C^{(i,j)}$ by assumption. \square

Lemma 3.7. *In co-path Hopf algebra $kQ^c(G, r, \overrightarrow{\rho}, u)$, assume $C := g^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$ and $a_{g,1}^{(i,j)} \cdot g = qa_{g^2,g}^{(i,j)}$. If i_1, i_2, \dots, i_m are non-negative integers, then*

$$a_{g^{i_m+1},g^{i_m}}^{(i,j)} \cdot a_{g^{i_{m-1}+1},g^{i_{m-1}}}^{(i,j)} \cdots a_{g^{i_1+1},g^{i_1}}^{(i,j)} = q^{\beta_m}(m)_q! P_{g^{\alpha_m}}^{(i,j)}(g, m)$$

where $\alpha_m = i_1 + i_2 + \dots + i_m$, $P_h^{(i,j)}(g, m) := a_{g^m h, g^{m-1}h}^{(i,j)} a_{g^{m-1}h, g^{m-2}h}^{(i,j)} \cdots a_{gh, h}^{(i,j)}$, $\beta_1 = 0$ and $\beta_m = \sum_{j=1}^{m-1} (i_1 + i_2 + \dots + i_j)$ if $m > 1$.

Proof. We prove the equality by induction on m . For $m = 1$, it is easy to see that the equality holds. Now suppose $m > 1$. We have

$$\begin{aligned}
& a_{g^{i_m+1}, g^{i_m}}^{(i,j)} \cdot a_{g^{i_{m-1}+1}, g^{i_{m-1}}}^{(i,j)} \cdots a_{g^{i_1+1}, g^{i_1}}^{(i,j)} \\
= & a_{g^{i_m+1}, g^{i_m}}^{(i,j)} \cdot (a_{g^{i_{m-1}+1}, g^{i_{m-1}}}^{(i,j)} \cdots a_{g^{i_1+1}, g^{i_1}}^{(i,j)}) \\
= & q^{\beta_{m-1}} (m-1)_q! a_{g^{i_m+1}, g^{i_m}}^{(i,j)} \cdot P_{g^{\alpha_{m-1}}}^{(i,j)}(g, m-1) \quad (\text{by inductive assumption}) \\
= & q^{\beta_{m-1}} (m-1)_q! a_{g^{i_m+1}, g^{i_m}}^{(i,j)} \cdot (a_{g^{\alpha_{m-1}+m-1}, g^{\alpha_{m-1}+m-2}}^{(i,j)} \cdots a_{g^{\alpha_{m-1}+1}, g^{\alpha_{m-1}}}^{(i,j)}) \\
= & q^{\beta_{m-1}} (m-1)_q! \sum_{l=1}^m [(g^{i_m+1} \cdot a_{g^{\alpha_{m-1}+m-1}, g^{\alpha_{m-1}+m-2}}^{(i,j)}) \cdots (g^{i_m+1} \cdot a_{g^{\alpha_{m-1}+l}, g^{\alpha_{m-1}+l-1}}^{(i,j)}) \\
& (a_{g^{i_m+1}, g^{i_m}}^{(i,j)} \cdot g^{\alpha_{m-1}+l-1}) (g^{i_m} \cdot a_{g^{\alpha_{m-1}+l-1}, g^{\alpha_{m-1}+l-2}}^{(i,j)}) \cdots (g^{i_m} \cdot a_{g^{\alpha_{m-1}+1}, g^{\alpha_{m-1}}}^{(i,j)})] \\
& (\text{by [CR02, Theorem 3.8]}) \\
= & q^{\beta_{m-1}} (m-1)_q! \sum_{l=1}^m [a_{g^{\alpha_m+m}, g^{\alpha_m+m-1}}^{(i,j)} \cdots a_{g^{\alpha_m+l+1}, g^{\alpha_m+l}}^{(i,j)} \\
& q^{\alpha_{m-1}+l-1} a_{g^{\alpha_m+l}, g^{\alpha_m+l-1}}^{(i,j)} a_{g^{\alpha_m+l-1}, g^{\alpha_m+l-2}}^{(i,j)} \cdots a_{g^{\alpha_m+1}, g^{\alpha_m}}^{(i,j)}] \quad (\text{by lemma 3.6}) \\
= & q^{\beta_{m-1}} (m-1)_q! \sum_{l=1}^m q^{\alpha_{m-1}+l-1} P_{g^{\alpha_m}}^{(i,j)}(g, m) \\
= & q^{\beta_{m-1}+\alpha_{m-1}} (m)_q! P_{g^{\alpha_m}}^{(i,j)}(g, m) \\
= & q^{\beta_m} (m)_q! P_{g^{\alpha_m}}^{(i,j)}(g, m). \quad \square
\end{aligned}$$

Recall that a braided algebra A in braided tensor category (\mathcal{C}, σ) with braiding σ is said to be braided commutative or quantum commutative, if $ab = \mu\sigma(a \otimes b)$ for any $a, b \in A$, where μ is the multiplication of A .

By [CR02, Example 3.11], the multiplication of any two arrows $a_{y,x}^{(i,j)}$ and $a_{w,v}^{(m,n)}$ in co-path Hopf algebra $kQ^c(G, r, \vec{\rho}, u)$ is

$$a_{y,x}^{(i,j)} \cdot a_{w,v}^{(m,n)} = (y \cdot a_{w,v}^{(m,n)})(a_{y,x}^{(i,j)} \cdot v) + (a_{y,x} \cdot w)(x \cdot a_{w,v}^{(m,n)}). \quad (3.5)$$

Lemma 3.8. *Let $C := x^G$, $D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$, $i' \in I_D(r, u)$, $j' \in J_D(i')$, $\alpha, \beta \in k$ with $a_{y,1}^{(i',j')} \cdot x = \alpha a_{yx,x}^{(i',j')}$ and $a_{x,1}^{(i,j)} \cdot y = \beta a_{xy,y}^{(i,j)}$ in co-path Hopf algebra $kQ^c(G, r, \vec{\rho}, u)$. If $xy = yx$ then $\alpha\beta = 1$ if and only if*

$$a_{x,1}^{(i,j)} \cdot a_{y,1}^{(i',j')} = \alpha^{-1} a_{y,1}^{(i',j')} \cdot a_{x,1}^{(i,j)} \quad (3.6)$$

Proof. By (3.5) and [ZCZ, Pro. 1.2], we have

$$\begin{aligned}
a_{x,1}^{(i,j)} \cdot a_{y,1}^{(i',j')} &= a_{xy,x}^{(i',j')} a_{x,1}^{(i,j)} + \beta a_{xy,y}^{(i,j)} a_{y,1}^{(i',j')}, \\
a_{y,1}^{(i',j')} \cdot a_{x,1}^{(i,j)} &= \alpha a_{yx,x}^{(i',j')} a_{x,1}^{(i,j)} + a_{yx,y}^{(i,j)} a_{y,1}^{(i',j')}.
\end{aligned} \quad (3.7)$$

Applying this we can complete the proof. \square

Lemma 3.9. *Assume that $\text{RSR}(G, r, \vec{\rho}, u)$ satisfies $C := \{g_C\} \subseteq Z(G)$ for any $C \in \mathcal{K}_r(G)$. Let $\rho_C^{(i)}(g_D) = q_{C,D}^{(i)} \text{id}$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$.*

(i) *The following conditions are equivalent:*

- (1) *$\text{RSR}(G, r, \vec{\rho}, u)$ is quantum symmetric*
- (2) *$q_{C,D}^{(i)} q_{D,C}^{(i')} = 1$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$.*

- (3) $a_{g_C,1}^{(i,j)} \cdot a_{g_D,1}^{(i',j')} = (q_{D,C}^{(i')})^{-1} a_{g_D,1}^{(i',j')} \cdot a_{g_C,1}^{(i,j)}$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in I_C(i)$, $j' \in J_C(i')$.
- (4) $\mathfrak{B}(G, r, \vec{\rho}, u)$ is quantum commutative in ${}^{kG}_{kG}\mathcal{YD}$.
- (5) $\mathfrak{B}(G, r, \vec{\rho}, u)$ is quatntum symmetric.
- (6) $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ is quantum symmetric.
- (ii) The following conditions are equivalent:
- (1) $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum weakly symmetric
- (2) $q_{C,D}^{(i)} q_{D,C}^{(i')} = 1$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$ with $(C, i) \neq (D, i')$.
- (3) $a_{g_C,1}^{(i,j)} \cdot a_{g_D,1}^{(i',j')} = (q_{D,C}^{(i')})^{-1} a_{g_D,1}^{(i',j')} \cdot a_{g_C,1}^{(i,j)}$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $i' \in I_D(r, u)$, $j \in I_C(i)$, $j' \in J_C(i')$ with $(C, i) \neq (D, i')$.

Proof. By [ZCZ, Lemma 2,2], $\text{diag}(kG[kQ_1^c, r, \vec{\rho}, u])$ is the Nichols algebra $\mathfrak{B}(G, r, \vec{\rho}, u)$ in ${}^{kG}_{kG}\mathcal{YD}$. By [ZCZ, Pro. 1.2],

$$\sigma^2(a_{g_C,1}^{(i,j)} \otimes a_{g_D,1}^{(i',j')}) = (q_{C,D}^{(i)} q_{D,C}^{(i')})^{-1} a_{g_C,1}^{(i,j)} \otimes a_{g_D,1}^{(i',j')}. \quad (3.8)$$

(i) By Lemma 3.3 (ii), (1), (5) and (6) are equivalent. It follows from (3.8) that (6) and (2) are equivalent. By Lemma 3.8, (3) and (2) are equivalent. Obviously (3) and (6) are equivalent. (3) and (4) are equivalent since $\mathfrak{B}(G, r, \vec{\rho}, u)$ is generated by kQ_1^1 .

(ii) It follows from (3.8) that (1) and (2) are equivalent. (2) and (3) are equivalent according to (3.6). \square

Lemma 3.10. (See [AS98, Lemma 3.3]) Let B be a Hopf algebra and R a braided Hopf algebra in ${}^B_B\mathcal{YD}$ with a linearly independent set $\{x_1, \dots, x_t\} \subseteq P(R)$, the set of all primitive elements in R . Assume that there exist $g_j \in G(B)$ (the set of all group-like elements in B) and $0 \neq k_{j,i} \in k$ such that

$$\delta(x_i) = g_i \otimes x_i, \quad g_i \cdot x_j = k_{ij} x_j, \quad \text{for all } i, j = 1, 2, \dots, t.$$

Then

$$\{x_1^{m_1} x_2^{m_2} \cdots x_t^{m_t} \mid 0 \leq m_j < N_j, 1 \leq j \leq t\}.$$

is linearly independent, where N_i is the order of $q_i := k_{ii}$ ($N_i = \infty$ when q_i is not a root of unit, or $q_i = 1$) for $1 \leq i \leq t$.

Proof. By the quantum binomial formula, if $1 \leq n_j < N_j$, then

$$\Delta(x_j^{n_j}) = \sum_{0 \leq i_j \leq n_j} \binom{n_j}{i_j}_{q_j} x_j^{i_j} \otimes x_j^{n_j - i_j}.$$

We use the following notation:

$$\mathbf{n} = (n_1, \dots, n_j, \dots, n_t), \quad x^{\mathbf{n}} = x_1^{n_1} \cdots x_j^{n_j} \cdots x_t^{n_t}, \quad |\mathbf{n}| = n_1 + \cdots + n_j + \cdots + n_t;$$

accordingly, $\mathbf{N} = (N_1, \dots, N_t)$, $\mathbf{1} = (1, \dots, 1)$. Also, we set

$$\mathbf{i} \leq \mathbf{n} \quad \text{if } i_j \leq n_j, j = 1, \dots, t.$$

Then, for $\mathbf{n} < \mathbf{N}$, we deduce from the quantum binomial formula that

$$\Delta(x^{\mathbf{n}}) = x^{\mathbf{n}} \otimes 1 + 1 \otimes x^{\mathbf{n}} + \sum_{0 \leq \mathbf{i} \leq \mathbf{n}, 0 \neq \mathbf{i} \neq \mathbf{n}} c_{\mathbf{n}, \mathbf{i}} x^{\mathbf{i}} \otimes x^{\mathbf{n} - \mathbf{i}}, \quad (3.9)$$

where $c_{\mathbf{n}, \mathbf{i}} \neq 0$ for all \mathbf{i} .

We shall prove by induction on r that the set

$$\{x^{\mathbf{n}} \mid |\mathbf{n}| \leq r, \mathbf{n} < \mathbf{N}\}$$

is linearly independent.

Let $r = 1$ and let $a_0 + \sum_{i=1}^t a_i x_i = 0$, with $a_j \in k$, $0 \leq j \leq t$. Applying ϵ , we see that $a_0 = 0$; by hypothesis we conclude that the other a_j 's are also 0.

Now let $r > 1$ and suppose that $z = \sum_{|\mathbf{n}| \leq r, \mathbf{n} < \mathbf{N}} a_{\mathbf{n}} x^{\mathbf{n}} = 0$. Applying ϵ , we see that $a_0 = 0$. Then

$$\begin{aligned} 0 &= \Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{1 \leq |\mathbf{n}| \leq r, \mathbf{n} < \mathbf{N}} a_{\mathbf{n}} \sum_{0 \leq \mathbf{i} \leq \mathbf{n}, 0 \neq \mathbf{i} \neq \mathbf{n}} c_{\mathbf{n}, \mathbf{i}} x^{\mathbf{i}} \otimes x^{\mathbf{n} - \mathbf{i}} \\ &= \sum_{1 \leq |\mathbf{n}| \leq r, \mathbf{n} < \mathbf{N}} \sum_{0 \leq \mathbf{i} \leq \mathbf{n}, 0 \neq \mathbf{i} \neq \mathbf{n}} a_{\mathbf{n}} c_{\mathbf{n}, \mathbf{i}} x^{\mathbf{i}} \otimes x^{\mathbf{n} - \mathbf{i}}. \end{aligned}$$

Now, if $|\mathbf{n}| \leq r$, $0 \leq \mathbf{i} \leq \mathbf{n}$, and $0 \neq \mathbf{i} \neq \mathbf{n}$, then $|\mathbf{i}| < r$ and $|\mathbf{n} - \mathbf{i}| < r$. By inductive hypothesis, the elements $x^{\mathbf{i}} \otimes x^{\mathbf{n} - \mathbf{i}}$ are linearly independent. Hence $a_{\mathbf{n}} c_{\mathbf{n}, \mathbf{i}} = 0$ and $a_{\mathbf{n}} = 0$ for all \mathbf{n} , $|\mathbf{n}| \geq 1$. Thus $a_{\mathbf{n}} = 0$ for all \mathbf{n} . \square

The quantum linear space was defined in [AS98, Lemma 3.4] and now is generalized as follows.

Definition 3.11. Let $0 \neq k_{i,j} \in k$ and $1 < N_i := \text{ord}(k_{k_{i,i}}) < \infty$ for any $i, j \in \Omega$, where Ω is a finite set. If R is the algebra generated by set $\{x_j \mid j \in \Omega\}$ with relations

$$x_i^{N_i} = 0, \quad x_i x_j = k_{i,j} x_j x_i \quad \text{for any } i, j \in \Omega \text{ with } i \neq j, \quad (3.10)$$

then R is called the generalized quantum linear space generated by $\{x_j \mid j \in \Omega\}$.

Definition 3.12. (i) $\text{RSR}(G, r, \vec{p}, u)$ is said to be a generalized quantum linear type if the following conditions are satisfied:

(GQL1) $xy = yx$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$.

(GQL2) there exists $k_{x,y}^{(i,j)} \in k$ such that $a_{x,1}^{(i,j)} \cdot y = k_{x,y}^{(i,j)} a_{xy,y}^{(i,j)}$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$.

(GQL3) $k_{x,y}^{(i,j)} k_{y,x}^{(i',j')} = 1$ for any $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$, $i' \in I_D(r, u)$, $j' \in J_D(i')$ with $(x, i, j) \neq (y, i', j')$.

(GQL4) $1 < N_x^{(i,j)} := \text{ord}(k_{x,x}^{(i,j)}) < \infty$ for any $C := x^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$.

(ii) $\text{RSR}(G, r, \vec{\rho}, u)$ is said to be a central quantum linear type if it is quantum symmetric and of the non-essentially infinite type with $C \subseteq Z(G)$ for any $C \in \mathcal{K}_r(G)$. In this case, $\mathfrak{B}(G, r, \vec{\rho}, u)$ is called a central quantum linear space over G .

Assume that A is an algebra with $\{b_\nu \mid \nu \in \Omega\} \subseteq A$ and \prec is a total order of Ω , $N_\nu \in \mathbb{N}$ or ∞ for any $\nu \in \Omega$. If

$$\{b_{\nu_1}^{m_1} b_{\nu_2}^{m_2} \cdots b_{\nu_n}^{m_n} \mid \nu_1 \prec \nu_2, \dots, \nu_n; 0 \leq m_s < N_{\nu_s}; 1 \leq s \leq n; n \in \mathbb{N}\} \quad (3.11)$$

is a basis of A , then the basis (3.11) is called a PBW basis generated by $\{b_\nu \mid \nu \in \Omega\}$. If $\{b_\nu \mid \nu \in \Omega\} \subseteq Q_1$, then it is called an arrow PBW basis.

It is well-known that every quantum linear space is a braided Hopf algebra and has a BPW basis (see [AS98, Lemma 3.4]). Of course, every generalized quantum linear space is finite dimensional. However, it is not known whether every generalized quantum linear space has an PBW basis.

Proposition 3.13. *If $\text{RSR}(G, r, \vec{\rho}, u)$ is of the generalized quantum linear type, then $\mathfrak{B}(G, r, \vec{\rho}, u)$ is a generalized quantum linear space with the arrow PBW basis*

$$\{b_{\nu_1}^{m_1} b_{\nu_2}^{m_2} \cdots b_{\nu_n}^{m_n} \mid \nu_1 \prec \nu_2, \dots, \nu_n; 0 \leq m_s < N_{\nu_s}; 1 \leq s \leq n; n \in \mathbb{N}\} \quad (3.12)$$

and

$$\dim(\mathfrak{B}(G, r, \vec{\rho}, u)) = \prod_{C:=x^G \in \mathcal{K}_r(G), i \in I_C(r, u), j \in J_C(i)} N_x^{(i,j)}, \quad (3.13)$$

where $\{b_\nu \mid \nu \in \Omega\} := Q_1^1$ with total order \prec and $N_{\nu_s} = N_x^{(i,j)} := \text{ord}(k_{x,x}^{(i,j)})$ if $b_{\nu_s} = a_{x,1}^{(i,j)}$.

Proof. Since any two different arrows in $\mathfrak{B}(G, r, \vec{\rho}, u)$ are quantum commutative (see Lemma 3.8) and $(b_{\nu_s})^{N_{\nu_s}} = 0$ (see Lemma 3.7), we have $\mathfrak{B}(G, r, \vec{\rho}, u)$ is generated by (3.12).

For any $\nu, \nu' \in \Omega$, $b_\nu = a_{x,1}^{(i,j)}$ and $b_{\nu'} = a_{y,1}^{(i',j')}$ with $C := x^G, D := y^G \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$, $i' \in I_D(r, u)$, $j' \in J_D(i')$, let $g_\nu = x$ and $k_{\nu,\nu'} = (k_{y,x}^{(i',j')})^{-1}$. By [ZCZ, Pro. 1.2] we have

$$\begin{aligned} \delta^-(b_\nu) &= \delta^-(a_{x,1}^{(i,j)}) = x \otimes a_{x,1}^{(i,j)} = g_\nu \otimes b_\nu \text{ and} \\ g_\nu \triangleright b_{\nu'} &= x \cdot a_{y,1}^{(i',j')} \cdot x^{-1} \\ &= x \cdot (k_{y,x}^{(i',j')})^{-1} a_{yx^{-1},x^{-1}}^{(i',j')} \text{ (by (GQL2))} \\ &= k_{\nu,\nu'} b_{\nu'} \text{ (by (GQL1)).} \end{aligned}$$

Therefore, by Lemma 3.10, (3.12) is linearly independent. Thus (3.12) is a basis of $\mathfrak{B}(G, r, \vec{\rho}, u)$.

Let R is the generalized quantum linear space generated by $\{b_\nu \mid \nu \in \Omega\} := kQ_1^1$. It is clear that there exists an algebra map ψ from R to $\mathfrak{B}(G, r, \vec{\rho}, u)$ by sending b_ν to b_ν for any $\nu \in \Omega$. Since $\mathfrak{B}(G, r, \vec{\rho}, u)$ has an arrow PBW basis (3.14), ψ is isomorphic. \square

Proposition 3.14. *Assume that $C = \{g_C\} \subseteq Z(G)$ and $\rho_C^{(i)}(g_D) = q_{C,D}^{(i)}$ id for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$. Then*

(i) $\text{RSC}(G, r, \vec{\chi}, u)$ is of the central quantum linear type if and only if $q_{C,D}^{(i)} q_{D,C}^{(j)} = 1$ and $1 < \text{ord}(q_{C,C}^{(i)}) < \infty$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in I_D(r, u)$.

(ii) $\text{RSC}(G, r, \vec{\chi}, u)$ is quantum weakly symmetric with non-essetially infinite type if and only if $q_{C,D}^{(i)} q_{D,C}^{(j)} = 1$ and $1 < \text{ord}(q_{C,C}^{(i)}) < \infty$ for any $C, D \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in I_D(r, u)$ with $(C, i) \neq (D, j)$.

Proof. (i) If $\text{RSC}(G, r, \vec{\chi}, u)$ is of the central quantum linear type, then $\dim \mathfrak{B}(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}) < \infty$ for any $C \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$. Let $N_C^{(i)} := \text{ord}(q_{C,C}^{(i)})$ ($N_C^{(i)} = \infty$ when $q_{C,C}^{(i)}$ is not a root of unit or $q_{C,C}^{(i)} = 1$). By Lemma 3.10, $\{(a_{g_C,1}^{(i,j)})^m \mid 0 \leq m < N_C^{(i)}\}$ is linearly independent. Thus $1 < \text{ord}(q_{C,C}^{(i)}) < \infty$. Since $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum symmetric, $q_{C,D}^{(i)} q_{D,C}^{(j)} = 1$ by Lemma 3.9.

Conversely, by Lemma 3.9, $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum symmetric. It is clear that $\text{RSR}(G, r, \vec{\rho}, u)$ is of the generalized quantum linear type. Thus it is of the non-essentially infinite type by Proposition 3.13.

(ii) It is similar to (i). \square

The following is the consequence of Proposition 3.13 and Proposition 3.15.

Proposition 3.15. *If $\text{RSR}(G, r, \vec{\rho}, u)$ is of the central quantum linear type, then $\mathfrak{B}(G, r, \vec{\rho}, u)$ is a generalized quantum linear space with the arrow BPW basis*

$$\{b_{\nu_1}^{m_1} b_{\nu_2}^{m_2} \cdots b_{\nu_n}^{m_n} \mid \nu_1 \prec \nu_2, \dots \prec \nu_n; 0 \leq m_s < N_{\nu_s}; 1 \leq s \leq n; n \in \mathbb{N}\} \quad (3.14)$$

and

$$\dim(\mathfrak{B}(G, r, \vec{\rho}, u)) = \prod_{C \in \mathcal{K}_r(G), i \in I_C(r, u)} (N_C^{(i)})^{\deg(\rho_C^{(i)})|C|}, \quad (3.15)$$

where $\{b_\nu \mid \nu \in \Omega\} := Q_1^1$ with total order \prec and $N_{\nu_s} = \text{ord}(q_{C,C}^{(i)})$ if $b_{\nu_s} = a_{g_C,1}^{(i,j)}$.

In particular, if $\text{RSR}(G, r, \vec{\rho}, u)$ is quantum weakly commutative and of -1 -type with $C \subseteq Z(G)$ for any $C \in \mathcal{K}_r(G)$, then it is of the central quantum linear type with $N_C^{(i)} = 2$ and

$$\dim(\mathfrak{B}(G, r, \vec{\rho}, u)) = 2^{\sum_{C \in \mathcal{K}_r(G), i \in I_C(r, u)} \deg(\rho_C^{(i)})|C|}. \quad (3.16)$$

Remark 3.16. $\text{RSR}(G, r, \vec{\rho}, u)$ is called a central ramification system with irreducible representations (or CRSR in short) if $C \subseteq Z(G)$ for any $C \in \mathcal{K}_r(G)$. If G is a real group and $r = r_C C$, then $\text{CRSR}(G, r, \vec{\rho}, u)$ is of finite type if and only if $\text{CRSR}(G, r, \vec{\rho}, u)$ is -1 -type. Indeed, The necessity follows from Proposition 2.6. the sufficiency follows from Proposition 3.14(i) since $q_{C,C}^{(i)} = -1$ for any $i \in I_C(u, r)$.

4 Program

In this section the programs to compute the representatives of conjugacy classes, centralizers of these representatives and character tables of these centralizers in Weyl groups of exceptional type are given.

By using the programs in GAP, papers [ZWCYb, ZWCYb] obtained the representatives of conjugacy classes of Weyl groups of exceptional type and all character tables of centralizers of these representatives. We use the results in [ZWCYb, ZWCYb] and the following program in GAP for Weyl group $W(E_6)$.

```
gap> L:=SimpleLieAlgebra("E",6,Rationals);;
gap> R:=RootSystem(L);;
gap> W:=WeylGroup(R);Display(Order(W));
gap > ccl:=ConjugacyClasses(W);;
gap> q:=NrConjugacyClasses(W);; Display (q);
gap> for i in [1..q] do
> r:=Order(Representative(ccl[i]));Display(r);;
> od; gap
> s1:=Representative(ccl[1]);cen1:=Centralizer(W,s1);;
gap> cl1:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[2]);cen1:=Centralizer(W,s1);;
gap> cl2:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[3]);cen1:=Centralizer(W,s1);;
gap> cl3:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[4]);cen1:=Centralizer(W,s1);;
gap> cl4:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[5]);cen1:=Centralizer(W,s1);;
gap> cl5:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[6]);cen1:=Centralizer(W,s1);;
gap> cl6:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[7]);cen1:=Centralizer(W,s1);;
gap> cl7:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[8]);cen1:=Centralizer(W,s1);;
```

```

gap> cl8:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[9]);cen1:=Centralizer(W,s1);
gap> cl9:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[10]);cen1:=Centralizer(W,s1);
gap> cl10:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[11]);cen1:=Centralizer(W,s1);
gap> cl11:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[12]);cen1:=Centralizer(W,s1);
gap> cl2:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[13]);cen1:=Centralizer(W,s1);
gap> cl13:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[14]);cen1:=Centralizer(W,s1);
gap> cl14:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[15]);cen1:=Centralizer(W,s1);
gap> cl15:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[16]);cen1:=Centralizer(W,s1);
gap> cl16:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[17]);cen1:=Centralizer(W,s1);
gap> cl17:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[18]);cen1:=Centralizer(W,s1);
gap> cl18:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[19]);cen1:=Centralizer(W,s1);
gap> cl19:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[20]);cen1:=Centralizer(W,s1);
gap> cl20:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[21]);cen1:=Centralizer(W,s1);
gap> cl21:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[22]);cen1:=Centralizer(W,s1);
gap> cl22:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[23]);cen1:=Centralizer(W,s1);
gap> cl23:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[24]);cen1:=Centralizer(W,s1);
> cl24:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[25]);cen1:=Centralizer(W,s1);
gap> cl25:=ConjugacyClasses(cen1);
gap> for i in [1..q] do
> s:=Representative(ccl[i]);cen:=Centralizer(W,s);
> char:=CharacterTable(cen);Display (cen);Display(char);

```

```

> od; gap> for i in [1..q] do
> s:=Representative(ccl[i]);cen:=Centralizer(W,s);;
> cl:=ConjugacyClasses(cen);;t:=NrConjugacyClasses(cen);;
> for j in [1..t] do
> if s=Representative(cl[j]) then
> Display(j);break; > fi;od;
> od;

```

The programs for Weyl groups of E_7 , E_8 , F_4 and G_2 are similar. It is possible that the order of representatives of conjugacy classes of G changes when one uses the program.

5 Tables about -1 - type

In this section all -1 - type bi-one Nichols algebras over Weyl groups of exceptional type up to graded pull-push YD Hopf algebra isomorphisms, are listed in table 1–12.

Table 1 is about Weyl group $W(E_6)$; Tables 2–4 are about Weyl group $W(E_7)$; Tables 5–10 are about Weyl group $W(E_8)$; Table 11 is about Weyl group $W(F_4)$; Table 12 is about Weyl group $W(G_2)$.

E_6					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\text{cl}_1[1]$	1		25	25
s_2	$\text{cl}_2[3]$	4	4,5,6,7,17	20	15
s_3	$\text{cl}_3[24]$	2	13,14,23,24,25	25	20
s_4	$\text{cl}_4[17]$	4	2,4,10,15,16	20	15
s_5	$\text{cl}_5[7]$	4	3,4,7,8	16	12
s_6	$\text{cl}_6[2]$	2	9,10,11,12,16,17,18,19,25	25	16
s_7	$\text{cl}_7[19]$	2	2,3,6,7,10,12,15,16,19,20	20	10
s_8	$\text{cl}_8[26]$	3		27	27
s_9	$\text{cl}_9[2]$	6	2,4,13	18	15
s_{10}	$\text{cl}_{10}[2]$	2	2,3,7,8,11,12,15,16,19,20,22	22	11
s_{11}	$\text{cl}_{11}[14]$	6	3,4,13	18	15
s_{12}	$\text{cl}_{12}[27]$	3		27	27
s_{13}	$\text{cl}_{13}[4]$	10	2	10	9
s_{14}	$\text{cl}_{14}[9]$	5		10	10
s_{15}	$\text{cl}_{15}[13]$	4	3,4,6,13,14	16	11
s_{16}	$\text{cl}_{16}[3]$	8	2	8	7
s_{17}	$\text{cl}_{17}[3]$	6	13	15	14
s_{18}	$\text{cl}_{18}[9]$	12	2	12	11
s_{19}	$\text{cl}_{19}[11]$	6	3,4	12	10
s_{20}	$\text{cl}_{20}[2]$	9		9	9
s_{21}	$\text{cl}_{21}[2]$	3		24	24
s_{22}	$\text{cl}_{22}[13]$	6	2,4,13	18	15
s_{23}	$\text{cl}_{23}[3]$	12	2	12	11
s_{24}	$\text{cl}_{24}[19]$	6	10,11,12	21	18
s_{25}	$\text{cl}_{25}[4]$	6	3,4,13	18	15

Table 1

E_7					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\text{cl}_1[1]$	1		60	60
s_2	$\text{cl}_2[16]$	18	2	18	17
s_3	$\text{cl}_3[15]$	9		18	18
s_4	$\text{cl}_4[2]$	3		48	48
s_5	$\text{cl}_5[4]$	6	2,3,4,8,11,12,14,40,43,44	48	38
s_6	$\text{cl}_6[2]$	2	2,4,6,8,10,12,15,16,18,20,22,26,27,28,30, 32,35,36,38,41,42,44,46,48,50,52,54,56,58,60	60	30
s_7	$\text{cl}_7[23]$	6	2,3,6,7,25,26	36	30
s_8	$\text{cl}_8[23]$	3		54	54
s_9	$\text{cl}_9[2]$	2	2,3,4,5,9,10,13,14,17,18,19,20,25,26,27, 28,33,35,36,39,40,43,44,45,46,51,52,53,54,55, 56,57,58,67,68,69,70,75,77,79,80,83,84,87,88	90	45
s_{10}	$\text{cl}_{10}[21]$	6	2,3,4,5,26,28	36	30
s_{11}	$\text{cl}_{11}[2]$	2	2,3,7,8,11,12,15,16,19,20,27,28,29,30,31, 32,37,38,39,40,42,44,47,48,51,52,55,56,58,61, 62,65,66,69,70,73,74	74	37
s_{12}	$\text{cl}_{12}[24]$	6	3,4,7,8,26,28	36	30
s_{13}	$\text{cl}_{13}[4]$	2	2,3,7,8,11,12,13,14,19,20,23,24,25,26, 27,28,37,38,39,40,42,43,45,46,49,50, 55,56,57,59,60,63,64,69,70,73,74	74	42
s_{14}	$\text{cl}_{14}[14]$	6	2,4,6,8,26,27,39,40,41,42	60	50
s_{15}	$\text{cl}_{15}[3]$	3		66	66
s_{16}	$\text{cl}_{16}[35]$	4	3,4,7,8,11,12,15,16,34,36,38,40,51,52, 55,56,59,60,63,64	80	60
s_{17}	$\text{cl}_{17}[2]$	2	25,26,27,28,61,62,71,72,73,74,75,76,77, 78,79,80,81,82,99,100,101,102,103,104,105,106	106	80
s_{18}	$\text{cl}_{18}[72]$	4	17,18,19,20,21,22,23,24,25,26,27,28,29, 30,31,32,33,34,35,36,45,46,47,48,49,50,51,52	76	48
s_{19}	$\text{cl}_{19}[2]$	2	25,26,27,28,29,30,31,32,43,44,45,46,47,48, 49,50,67,68,69,70,71,72,73,74,77,78,87,88,89,90	90	60
s_{20}	$\text{cl}_{20}[4]$	8	2,4,6,8	32	28
s_{21}	$\text{cl}_{21}[3]$	4	5,6,7,8,10,12,33,34,35,36,37,38,39,40,41, 42,43,44,50,52	60	40
s_{22}	$\text{cl}_{22}[8]$	12	2,5,7,8	48	44
s_{23}	$\text{cl}_{23}[34]$	6	49,50,51,52	60	56

Table 2

E_7					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{24}	$\text{cl}_{24}[47]$	4	2,5,7,8,10,13,15,16,34,35,38,39,51,52,53, 54,59,60,61,62	80	60
s_{25}	$\text{cl}_{25}[15]$	8	3,4,7,8	32	28
s_{26}	$\text{cl}_{26}[2]$	2	2,3,4,5,6,7,8,9,21,22,23,24,27,28, 37,38,39,40,41,42,43,44,53,54,55,56,57,58, 59,60,62,67,68,69,70,73,74,79,80,81,82,87, 88,89,90,95,96,97,98,101,102,104,106	106	53
s_{27}	$\text{cl}_{27}[51]$	6	2,3,6,7,27,28,37,38,39,40	60	50
s_{28}	$\text{cl}_{28}[36]$	4	2,3,6,7,10,11,14,15,34,36,37,39,49, 50,53,54,59,60,63,64	80	60
s_{29}	$\text{cl}_{29}[2]$	2	2,3,4,5,6,7,8,9,19,20,23,24,29,30,31, 32,37,38,39,40,45,46,47,48,53,54,55,56, 61,62,63,64,69,70,71,72,75,76,79,80	80	40
s_{30}	$\text{cl}_{30}[2]$	2	2,3,4,5,6,7,8,9,21,22,23,24,33,34,35, 36,37,38,39,40,49,50,51,52,53,54,55,56, 65,66,67,68,69,70,71,72,77,78,79,80	80	40
s_{31}	$\text{cl}_{31}[22]$	4	2,3,4,5,10,11,12,13,21,22,23,24,29,30, 31,32,34,36,38,40,41,42,45,46,49,50,53,54	76	48
s_{32}	$\text{cl}_{32}[30]$	6	2,7,8,20,29,30,38	42	35
s_{33}	$\text{cl}_{33}[30]$	6	3,4,7,8,26,28	36	30
s_{34}	$\text{cl}_{34}[80]$	4	2,3,4,5,10,11,12,13,34,35,38,39,51,52, 53,54,59,60,61,62	80	60
s_{35}	$\text{cl}_{35}[14]$	12	2,4,6,8	48	44
s_{36}	$\text{cl}_{36}[40]$	6	3,5,6,8,11,13,14,16,49,52	60	50
s_{37}	$\text{cl}_{37}[49]$	6	2,3,4,5,26,27,37,38,43,44	60	50
s_{38}	$\text{cl}_{38}[50]$	6	2,5,7,8,27,28,31,32,50	54	45
s_{39}	$\text{cl}_{39}[7]$	6	2,3,6,7	24	20
s_{40}	$\text{cl}_{40}[2]$	10	2,3	20	18
s_{41}	$\text{cl}_{41}[21]$	5		30	30
s_{42}	$\text{cl}_{42}[33]$	12	3,4,7,8	48	44
s_{43}	$\text{cl}_{43}[39]$	6	19,20,21,22,27,28	42	36
s_{44}	$\text{cl}_{44}[5]$	4	3,5,7,9,10,12,14,16,19,21,23,25,26,28,30,32	64	48
s_{45}	$\text{cl}_{45}[6]$	6	2,3,15,16,19,20,39,40,51,52,62	66	55
s_{46}	$\text{cl}_{46}[6]$	6	2,3,6,7	24	20
s_{47}	$\text{cl}_{47}[5]$	10	2,4	20	18
s_{48}	$\text{cl}_{48}[12]$	10	2,4,21	30	27

Table 3

E_7					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{49}	$\text{cl}_{49}[5]$	30	2	30	29
s_{50}	$\text{cl}_{50}[15]$	15		30	30
s_{51}	$\text{cl}_{51}[8]$	7		14	14
s_{52}	$\text{cl}_{52}[2]$	14	2	14	13
s_{53}	$\text{cl}_{53}[53]$	6	3,4,7,8,26,28,39,40,43,44	60	50
s_{54}	$\text{cl}_{54}[5]$	8	3,5,6,8	32	28
s_{55}	$\text{cl}_{55}[10]$	12	3,4	24	22
s_{56}	$\text{cl}_{56}[14]$	8	3,5,6,8	32	28
s_{57}	$\text{cl}_{57}[3]$	4	2,3,5,6,10,12,21,22,23,24,31,32,37,38,39, 40,41,42,50,52	60	40
s_{58}	$\text{cl}_{58}[15]$	12	2,4	24	22
s_{59}	$\text{cl}_{59}[38]$	12	3,5,6,8	48	44
s_{60}	$\text{cl}_{60}[9]$	4	2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32	64	48

Table 4

E_8					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\text{cl}_1[1]$	1		112	112
s_2	$\text{cl}_2[29]$	30	2	30	29
s_3	$\text{cl}_3[23]$	15		30	30
s_4	$\text{cl}_4[2]$	5		45	45
s_5	$\text{cl}_5[3]$	3		102	102
s_6	$\text{cl}_6[6]$	10	6,7,27,41	45	41
s_7	$\text{cl}_7[2]$	2	3,4,11,12,16,17,18,19,29,30,32,33,34, 37,38,45,46,51,52,56,57,60,63,64,65,66,71,79, 80,82,83,89,90,91,92,95,96,99,100,103,104, 106,107,108,112	112	67
s_8	$\text{cl}_8[4]$	6	4,5,31,33,34,35,36,61,62,79,81,82,92,97	102	88
s_9	$\text{cl}_9[4]$	30	2,4	60	58
s_{10}	$\text{cl}_{10}[13]$	15		60	60
s_{11}	$\text{cl}_{11}[5]$	5		70	70
s_{12}	$\text{cl}_{12}[2]$	3		150	150
s_{13}	$\text{cl}_{13}[46]$	10	2,4,6,8,41,44	60	54
s_{14}	$\text{cl}_{14}[3]$	2	2,4,6,8,10,12,14,16,18,20,22,24,27,28,31,32, 34,36,38,40,42,44,48,49,50,54,55,56,58,60,62,64, 67,68,71,72,74,76,79,80,83,84,86,88,90,92,94,96, 98,100,102,104,106,108,110,112,114,116,118,120	120	60
s_{15}	$\text{cl}_{15}[8]$	6	2,3,6,7,29,30,31,32,37,38,39,40,77,78, 79,80,101,102,103,104,123,124	132	110
s_{16}	$\text{cl}_{16}[16]$	30	2,3	60	58
s_{17}	$\text{cl}_{17}[9]$	10	2,3,23,24,43,44,62	70	63
s_{18}	$\text{cl}_{18}[4]$	6	2,3,15,16,26,35,36,37,38,57,58,60,75,76, 87,88,99,100,102,117,118,128,135,136,146	150	125
s_{19}	$\text{cl}_{19}[23]$	20	3,4	40	38
s_{20}	$\text{cl}_{20}[40]$	10	41,42	50	48
s_{21}	$\text{cl}_{21}[8]$	2	9,10,43,44,53,54,55,56,77,78,87,88,89,90, 91,92,109,110,111,112,121,134,151,152,153,154, 155,156,157,158,159,162,163,164,165,166,167	167	130
s_{22}	$\text{cl}_{22}[3]$	4	2,3,4,5,19,20,23,24,35,36,39,40,43,44,47, 48,66,68,77,78,79,80,85,86,87,88,90,92,115, 116,119,120,131,132,135,136	144	108
s_{23}	$\text{cl}_{23}[39]$	10	3,4,7,8,42,44	60	54

Table 5

E_8					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{24}	$\text{cl}_{24}[2]$	2	2,3,7,8,11,12,15,16,19,20,23,24,29,30,31,32, 35,36,39,40,43,44,51,52,53,54,55,56,59,60,63,64,69, 70,71,72,75,76,81,82,83,84,87,88,91,92,95,96,99, 100,103,104,107,108,111,112,115,116,119,120	120	60
s_{25}	$\text{cl}_{25}[44]$	10	3,5,6,8,41	50	45
s_{26}	$\text{cl}_{26}[2]$	2	27,28,29,30,31,32,33,34,55,56,79,80,81,82, 83,84,85,86,89,90,91,92,93,94,95,96,97,98,99,100, 101,102,103,104,105,106,107,108,122,123,124,125, 126,127,128,129,130,131,132,133,134,151,152,153, 154,157,158,159,160,161,162,163	167	105
s_{27}	$\text{cl}_{27}[28]$	20	2,3	40	38
s_{28}	$\text{cl}_{28}[4]$	4	2,3,4,5,19,20,23,24,37,38,39,40,45,46,47, 48,66,68,78,79,80,81,82,88,89,90,91,92,115,116, 119,120,131,132,135,136	144	108
s_{29}	$\text{cl}_{29}[3]$	4	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,66, 68,70,72,73,75,77,79,97,98,101,102,105,106,109, 110,115,116,119,120,123,124,127,128	160	120
s_{30}	$\text{cl}_{30}[3]$	4	2,3,4,5,18,19,27,28,29,30,33,34,35,36,57, 58,59,60,73,76	80	60
s_{31}	$\text{cl}_{31}[72]$	6	25,26,31,32,39,40,45,46,68	72	63
s_{32}	$\text{cl}_{32}[2]$	3		135	135
s_{33}	$\text{cl}_{33}[58]$	8	3,4,7,8,34,35,49,50,55,56	80	70
s_{34}	$\text{cl}_{34}[3]$	4	5,6,7,8,11,12,15,16,29,30,31,32,38,39,40,41, 42,55,56,57,58,63,64,65,66,85,86,87,88,89,90,91,92, 98,99,100,101,102,108,119,120,131,132,135,136	140	95
s_{35}	$\text{cl}_{35}[2]$	2	31,32,33,34,35,36,37,38,63,64,65,66,67,68, 69,70,71,72,73,74,104,105,106,107,108,109,110,111, 112,113,114,115,118,119,130,131,132,133,146,147, 148,149,150,151,152,153,154,155,156,157,159,168, 169,170,171,172,173,174,175,178,179,182,183,198, 199,200,201,202,203,208,209,210,211,213,215	215	140
s_{36}	$\text{cl}_{36}[64]$	8	2,4,6,8,34,36,51,52,55,56	80	70
s_{37}	$\text{cl}_{37}[44]$	4	9,10,11,12,13,14,15,16,17,18,21,22,43,44	44	30

Table 6

E_8					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{38}	$\text{cl}_{38}[2]$	2	21,22,23,24,25,26,27,28,33,34,35,36,41, 42,43,44,69,70,71,72,75,76,79,80,89,90,91,92, 93,94,95,96,97,98,99,100,101,102,103,104	105	65
s_{39}	$\text{cl}_{39}[73]$	4	4,5,8,9,10,11,14,15,20,21,24,25,26,27, 30,31,66,67,70,71,75,76,77,78,81,82,87,88	112	84
s_{40}	$\text{cl}_{40}[13]$	24	2,3	48	46
s_{41}	$\text{cl}_{41}[7]$	12	5,6,7,8,27,28,65,66,71,72	96	86
s_{42}	$\text{cl}_{42}[4]$	6	49,50,51,52,53,54,55,56,91,92,93,94,95, 96,97,98,145,146	150	132
s_{43}	$\text{cl}_{43}[98]$	12	4,5,6,7,12,13,14,15,98,100	120	110
s_{44}	$\text{cl}_{44}[4]$	6	57,58,59,60,133,134,139,140,141,142	150	140
s_{45}	$\text{cl}_{45}[3]$	4	5,6,7,8,11,12,15,16,29,30,31,32,33,34,35, 36,38,55,56,57,58,61,62,63,64,85,86,87,88,89,90, 91,92,93,94,95,96,98,108,119,120,131,132,133,134	140	95
s_{46}	$\text{cl}_{46}[19]$	8	2,4,6,8,9, 10,12,14,16	64	56
s_{47}	$\text{cl}_{47}[3]$	4	33,34,35,36,37,38,39,40,41,42,43,44,45,46, 47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62, 63,64,69,70,71,72,77,78,79,80,81,82,83,84,85,86, 87,88,137,138,139,140,141,142,143,144,146,148, 151,152,155,156,159,160	178	114
s_{48}	$\text{cl}_{48}[4]$	6	2,3,4,5,6,7,8,9,51,52,55,56,59,60,89,90,91, 92,97,98,99,100,134,141,142	150	125
s_{49}	$\text{cl}_{49}[4]$	8	4,5,6,7,33,34,35,36,66,67	80	70
s_{50}	$\text{cl}_{50}[3]$	4	2,3,4,5,9,10,11,12,18,20,22,24,41,42,43,44, 49,50,57,58,59,60,63,64,69,70,71,72,75,76,77,78, 79,80,85,86,98,100,102,104	120	80
s_{51}	$\text{cl}_{51}[3]$	4	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,66, 67,70,71,73,76,77,80,97,98,103,104,105,106,111, 112,115,116,117,118,123,124,125,126	160	120
s_{52}	$\text{cl}_{52}[9]$	6	3,4,6,9,10,13,15,16,49,52,54,55	72	60

Table 7

E_8					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{53}	$\text{cl}_{53}[2]$	2	2,3,4,5,6,7,8,9,19,20,23,24,27,28,31,32, 37,38,39,40,45,46,47,48,53,54,55,56,61,62,63,64, 66,68,71,72,75,76,79,80,83,84,89,90,91,92,97,98, 99,100,105,106,107,108,113,114,115,116,121, 122,123,124,129,130,131,132,137,138,139,140, 145,146,147,148,150,152,154,156,159,160,163, 164,167,168,171,172,175,176,179,180	180	90
s_{54}	$\text{cl}_{54}[19]$	12	2,4,6,8	48	44
s_{55}	$\text{cl}_{55}[4]$	6	37,38,39,40,43,44,45,46,59,60,61,62,103, 104,105,106,113,114	126	108
s_{56}	$\text{cl}_{56}[2]$	3		144	144
s_{57}	$\text{cl}_{57}[4]$	6	2,3,5,6,7,8,15,16,18,23,24,25,26,29,30,33,34, 60,63,64,66,105,106,111,112,113,114,136,139,140	144	114
s_{58}	$\text{cl}_{58}[32]$	6	2,4,7,9,11,13,14,16,53,54,55,56,61,62, 63,64,99,100	108	90
s_{59}	$\text{cl}_{59}[42]$	8	3,5,6,8,11,13,14,16	64	56
s_{60}	$\text{cl}_{60}[10]$	6	6,7,8,9,10,11,12,13	48	40
s_{61}	$\text{cl}_{61}[70]$	6	2,3,4,5,10,11,12,13,50,52,54,56	72	60
s_{62}	$\text{cl}_{62}[2]$	2	2,3,4,5,6,7,8,9,21,22,23,24,29,30,31,32,41, 42,43,44,45,46,47,48,57,58,59,60,61,62,63,64,67, 68,73,74,75,76,81,82,83,84,93,94,95,96,97,98,99, 100,117,118,119,120,121,122,123,124,125,126, 127,128,129,130,131,132,141,142,143,144,145, 146,147,148,151,152,155,156,161,162,163, 164,169,170,171,172,177,178,179,180	180	90
s_{63}	$\text{cl}_{63}[4]$	6	25,26,33,34,51,52,53,54,68,77,78,79,80, 109,112,113,130,131	135	117
s_{64}	$\text{cl}_{64}[36]$	6	2,3,15,16,21,22,38,39,47,48,53,54,74,75	84	70
s_{65}	$\text{cl}_{65}[28]$	18	2,4,37	54	51
s_{66}	$\text{cl}_{66}[26]$	9		54	54
s_{67}	$\text{cl}_{67}[18]$	18	3,4	36	34
s_{68}	$\text{cl}_{68}[4]$	6	2,3,5,6,7,8,15,16,21,22,23,24,27,28,79, 80,85,86,87,88	96	76
s_{69}	$\text{cl}_{69}[12]$	18	2,4	36	34
s_{70}	$\text{cl}_{70}[4]$	6	2,3,5,6,7,8,14,15,19,20,21,22,26,27,78, 79,83,84,85,86	96	76

Table 8

E_8					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{71}	$\text{cl}_{71}[18]$	9		54	54
s_{72}	$\text{cl}_{72}[24]$	18	2,5,6	54	51
s_{73}	$\text{cl}_{73}[40]$	12	4,5,6,7	48	44
s_{74}	$\text{cl}_{74}[3]$	4	2,3,4,5,9,10,11,12,17,20,22,23,41,42, 43,44,49,50,57,58,59,60,63,64,69,70,71,72, 75,76,77,78,79,80,85,86,97,100,102,103	120	80
s_{75}	$\text{cl}_{75}[7]$	12	2,3,5,6,27,28,61,62,65,66	96	86
s_{76}	$\text{cl}_{76}[12]$	12	2,3,6,7,49,52	72	66
s_{77}	$\text{cl}_{77}[36]$	12	2,9,10,38,47,48,74	84	77
s_{78}	$\text{cl}_{78}[4]$	6	29,56,57,58,59,60,61,62,63,100,101, 102,103,104,105,106,107,108,127,129,130	147	126
s_{79}	$\text{cl}_{79}[43]$	12	7,8,14,37,38	48	43
s_{80}	$\text{cl}_{80}[4]$	4	11,12,17,18,19,20,26,35,36,39,40,41, 46,47,48,49	59	43
s_{81}	$\text{cl}_{81}[15]$	20	2	20	19
s_{82}	$\text{cl}_{82}[7]$	12	2,3,4,5,50,52,75,76,79,80	120	110
s_{83}	$\text{cl}_{83}[3]$	4	2,3,4,5,6,7,8,9,35,36,39,40,43,44,47, 48,69,70,71,72,77,78,79,80,85,86,87,88,93, 94,95,96,130,132,139,140,143,144,147,148, 151,152,173,174,175,176,181,182,183,184	200	150
s_{84}	$\text{cl}_{84}[107]$	12	4,5,6,7,12,13,14,15,98,100	120	110
s_{85}	$\text{cl}_{85}[11]$	14	2,3	28	26
s_{86}	$\text{cl}_{86}[26]$	7		28	28
s_{87}	$\text{cl}_{87}[3]$	14	3,4	28	26
s_{88}	$\text{cl}_{88}[18]$	14	2,4	28	26
s_{89}	$\text{cl}_{89}[100]$	6	3,5,7,9,10,12,14,16,53,54,55,56,61,62, 63,64,99,100	108	90
s_{90}	$\text{cl}_{90}[56]$	6	3,4,23,24,29,30,38,39,43,44,57,58,73,76	84	70

Table 9

E_8					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{91}	$\text{cl}_{91}[29]$	8	2,3,10,23,24	32	27
s_{92}	$\text{cl}_{92}[8]$	24	2	24	23
s_{93}	$\text{cl}_{93}[9]$	12	2,4,6,8,50,52	72	66
s_{94}	$\text{cl}_{94}[4]$	6	2,3,5,6,7,8,38,41,42,51,52,75,76,77,78, 93,94,104,113,114,122	126	105
s_{95}	$\text{cl}_{95}[35]$	12	2,4,17,18,29,30	72	66
s_{96}	$\text{cl}_{96}[60]$	6	31,32,33,34,67,68	72	66
s_{97}	$\text{cl}_{97}[7]$	12	2,3,4,5,50,52,75,76,79,80	120	110
s_{98}	$\text{cl}_{98}[19]$	12	2,4	24	22
s_{99}	$\text{cl}_{99}[59]$	12	6,7,8,9,10,11,12,13	96	88
s_{100}	$\text{cl}_{100}[4]$	6	2,3,4,5,6,7,8,9,50,52,54,56,75,76,79,80, 83,84,87,88	120	100
s_{101}	$\text{cl}_{101}[80]$	12	4,5,8,9,10,11,14,15	96	88
s_{102}	$\text{cl}_{102}[84]$	6	2,3,6,7,10,11,14,15,51,52,53,54,59,60, 61,62,98,99	108	90
s_{103}	$\text{cl}_{103}[82]$	12	2,9,10,37,39,40,74	84	77
s_{104}	$\text{cl}_{104}[54]$	8	3,5,6,8,17,18,25,26,65,68	80	70
s_{105}	$\text{cl}_{105}[35]$	30	2,4	60	58
s_{106}	$\text{cl}_{106}[6]$	6	2,3,4,5,27,28,31,32,35,36,39,40,75,76, 79,80,99,100,103,104,122,124	132	110
s_{107}	$\text{cl}_{107}[11]$	8	3,4	16	14
s_{108}	$\text{cl}_{108}[4]$	6	2,3,4,5,27,28,31,32,53,54,55,56,58,81, 82,95,96,97,98,119,120,121,122,140,146	150	125
s_{109}	$\text{cl}_{109}[4]$	6	2,3,4,5,6,7,8,9,49,50,51,52,73,74,75,76, 77,78,79,80	120	100
s_{110}	$\text{cl}_{110}[2]$	24	3,4	48	46
s_{111}	$\text{cl}_{111}[23]$	12	2,4,26	36	33
s_{112}	$\text{cl}_{112}[92]$	6	3,5,6,8,23,24,35,36,49,50,61,62,73,76, 77,78,101,102	108	90

Table 10

F_4					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\text{cl}_1[1]$	1		25	25
s_2	$\text{cl}_2[2]$	2	9,10,11,12,16,17,18,19,25	25	16
s_3	$\text{cl}_3[25]$	2	17,18,19,20,25	25	20
s_4	$\text{cl}_4[16]$	4	3,4,6,13,14	16	11
s_5	$\text{cl}_5[5]$	3		18	18
s_6	$\text{cl}_6[15]$	6	3,4,13	18	15
s_7	$\text{cl}_7[10]$	2	2,3,4,5,10,12,15,16,19,20	20	10
s_8	$\text{cl}_8[19]$	2	2,4,6,8,9,10,13,14,15,16	20	10
s_9	$\text{cl}_9[16]$	4	3,5,6,8	16	12
s_{10}	$\text{cl}_{10}[16]$	3		18	18
s_{11}	$\text{cl}_{11}[12]$	6	3,4,13	18	15
s_{12}	$\text{cl}_{12}[9]$	3		21	21
s_{13}	$\text{cl}_{13}[21]$	6	10,11,12	21	18
s_{14}	$\text{cl}_{14}[11]$	12	2	12	11
s_{15}	$\text{cl}_{15}[10]$	6	2,3	12	10
s_{16}	$\text{cl}_{16}[12]$	6	2,4	12	10
s_{17}	$\text{cl}_{17}[19]$	2	2,3,4,6,7,10,12,15,16,19,20	20	10
s_{18}	$\text{cl}_{18}[18]$	2	2,4,6,8,9,12,13,14,19,20	20	10
s_{19}	$\text{cl}_{19}[6]$	4	2,4,6,8	16	12
s_{20}	$\text{cl}_{20}[11]$	6	2,3	12	10
s_{21}	$\text{cl}_{21}[12]$	6	2,4	12	10
s_{22}	$\text{cl}_{22}[2]$	2	6,7,8,9,10,11,12,13	16	8
s_{23}	$\text{cl}_{23}[4]$	8	2	8	7
s_{24}	$\text{cl}_{24}[17]$	4	2,4,6,8,18	20	15
s_{25}	$\text{cl}_{25}[9]$	4	2,4,6,8,17	20	15

Table 11

G_2					
s_i	$\text{cl}_i[p]$	$\text{Order}(s_i)$	the j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\text{cl}_1[1]$	1		6	6
s_2	$\text{cl}_2[3]$	2	2,4	4	2
s_3	$\text{cl}_3[3]$	2	2,4	4	2
s_4	$\text{cl}_4[4]$	2	3,4,5	6	3
s_5	$\text{cl}_5[3]$	6	2	6	5
s_6	$\text{cl}_6[5]$	3		6	6

Table 12

6 Bi-one Nichols algebras over Weyl groups of exceptional type

In this section all -1 -type bi-one Nichols algebra over Weyl groups G of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are given.

In Table 1–12, we use the following notations. s_i denotes the representative of i -th conjugacy class of G (G is the Weyl group of exceptional type); $\chi_i^{(j)}$ denotes the j -th character of G^{s_i} for any i ; $\nu_i^{(1)}$ denotes the number of conjugacy classes of the centralizer G^{s_i} ; $\nu_i^{(2)}$ denote the number of character $\chi_i^{(j)}$ of G^{s_i} with non -1 -type $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$; $\text{cl}_i[j]$ denote that s_i is in j -th conjugacy class of G^{s_i} .

We give one of the main results.

Theorem 1. *Let G be a Weyl group of exceptional type. Then*

(i) *For any bi-one Nichols algebra $\mathfrak{B}(\mathcal{O}_s, \chi)$ over Weyl group G , there exist s_i in the first column of the table of G and j with $1 \leq j \leq \nu_i^{(1)}$ such that $(kG, \mathfrak{B}(\mathcal{O}_s, \chi)) \cong (kG, \mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)}))$ as graded pull-push YD Hopf algebras;*

(ii) *$\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of -1 -type if and only if j appears in the fourth column of the table of G ;*

(iii) *$\dim(\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})) = \infty$ if j does not appears in the fourth column of the table of G .*

Proof. (i) We assume that G is the Weyl group of E_6 without loss of generality. There exists s_i such that s_i and s are in the same conjugacy class since s_1, s_2, \dots, s_{25} are the representatives of all conjugacy classes of G . Lemma 1.1 and [ZCZ, The remark of Pro. 1.5] or Proposition 1.5 yield that there exists j such that $(kG, \mathfrak{B}(\mathcal{O}_s, \chi)) \cong (kG, \mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)}))$ as graded pull-push YD Hopf algebras, since $\chi_i^{(1)}, \chi_i^{(2)}, \dots, \chi_i^{\nu_i^{(1)}}$ are all characters of all irreducible representations of G^{s_i} .

(ii) It follows from the program.

(iii) It follows from Lemma 1.3. \square

By [Ca72], $W(G_2)$ is isomorphic to dihedral group D_6 . Set $y = s_5$ and $x = s_3$. It is clear that $xyx = y^{-1}$ with $\text{ord}(y) = 6$ and $\text{ord}(x) = 2$. Thus it follows from [AF07, Table 2] that $\dim(\mathfrak{B}(\mathcal{O}_{s_5}, \chi_5^{(2)})) = 4 < \infty$.

It is clear that if there exists $\phi \in \text{Aut}(G)$ such that $\phi(s_i) = s_j$ then $\text{ord}(s_i) = \text{ord}(s_j)$, $\nu_i^{(1)} = \nu_j^{(1)}$, $\nu_i^{(2)} = \nu_j^{(2)}$ for Weyl group G of exceptional type. Consequently, the representative system of iso-conjugacy classes of $W(E_6)$ is $\{s_i \mid 1 \leq i \leq 25\}$. The representative system of iso-conjugacy classes of $W(F_4)$ is $\{s_i \mid 1 \leq i \leq 25, i \neq 8, 10, 11, 16, 17, 18, 19, 20, 21, 25\}$. The representative system of iso-conjugacy classes of $W(G_2)$ is $\{s_1, s_2, s_4, s_5, s_6\}$.

7 Pointed Hopf algebras over Weyl groups of exceptional type

In this section all central quantum linear spaces over Weyl groups of exceptional type are found.

Lemma 7.1. $Z(W(E_6)) = \{1\}$; $Z(W(E_7)) = \{1, s_6\}$; $Z(W(E_8)) = \{1, s_7\}$; $Z(W(F_4)) = \{1, s_2\}$; $Z(W(G_2)) = \{1, s_4\}$.

Proof. If $s_i \in Z(G)$, then $G^{s_i} = G$.

(i) Let $G = W(W_6)$. The number of conjugacy classes of G is 25 by table 1. The numbers of conjugacy classes of both G^{s_3} and G^{s_6} also are 25. G , G^{s_3} and G^{s_6} have 16, 8 and 4 one dimensional representations, respectively, according to the character tables in [ZWCYb]. Thus s_3 and s_6 do not belong to the center of G .

(ii) Let $G = W(W_7)$. The number of conjugacy classes of G , G^{s_6} , $G^{s_{14}}$, $G^{s_{21}}$, $G^{s_{23}}$, $G^{s_{27}}$, $G^{s_{36}}$, $G^{s_{37}}$, $G^{s_{53}}$, $G^{s_{57}}$ is 60 by table 1–4. They have 2, 2, 24, 8, 3, 24, 48, 24, 24 and 8 one dimensional representations, respectively, according to the character tables in [ZWCYb]. Thus they do not belong to the center of G but s_6 . Obviously $s_6 \in Z(G)$.

(iii) Let $G = W(W_8)$. The number of conjugacy classes of G , G^{s_7} , and $G^{s_{39}}$ is 112 by table 5–10. They have 2, 2 and 64 one dimensional representations, respectively, according to the character tables in [ZWCYb]. Thus s_{39} does not belong to the center of G . Obviously $s_7 \in Z(G)$.

(iv) Let $G = W(F_4)$. The number of conjugacy classes of G , G^{s_2} , and G^{s_3} is 25 by table 11. They have 4, 4 and 16 one dimensional representations, respectively, according to the character tables in [ZWCYb]. Thus s_3 does not belong to the center of G . Obviously $s_2 \in Z(G)$.

(v) Let $G = W(G_2)$. The number of conjugacy classes of G , G^{s_4} , G^{s_5} , and G^{s_6} is 6 by table 12. They have 4, 4, 6 and 6 one dimensional representations, respectively, according

to the character tables in [ZWCYa]. Thus s_5 and s_6 do not belong to the center of G . Obviously $s_4 \in Z(G)$. \square

We give the other main result.

Theorem 2. *Every central quantum linear space $\mathfrak{B}(G, r, \overrightarrow{\rho}, u)$ over Weyl Groups of exceptional type is one case in the following:*

- (i) $G = W(E_7)$, $C = \{s_6\}$, $r = r_C C$ and $\chi_C^{(i)} \in \{\chi_6^{(j)} \mid j = 2, 4, 6, 8, 10, 12, 15, 16, 18, 20, 22, 26, 27, 28, 30, 32, 35, 36, 38, 41, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60\}$ for any $i \in I_C(r, u)$.
- (ii) Let $G = W(E_8)$, $C = \{s_7\}$, $r = r_C C$ and $\chi_C^{(i)} \in \{\chi_7^{(j)} \mid j = 3, 4, 11, 12, 16, 17, 18, 19, 29, 30, 32, 33, 34, 37, 38, 45, 46, 51, 52, 56, 57, 60, 63, 64, 65, 66, 71, 79, 80, 82, 83, 89, 90, 91, 92, 95, 96, 99, 100, 103, 104, 106, 107, 108, 112\}$ for any $i \in I_C(r, u)$.
- (iii) Let $G = W(F_4)$, $C = \{s_2\}$, $r = r_C C$ and $\chi_C^{(i)} \in \{\chi_2^{(j)} \mid j = 9, 10, 11, 12, 16, 17, 18, 19, 25\}$ for any $i \in I_C(r, u)$.
- (iv) Let $G = W(G_2)$, $C = \{s_4\}$, $r = r_C C$ and $\chi_C^{(i)} \in \{\chi_4^{(3)}, \chi_4^{(4)}, \chi_4^{(5)}\}$ for any $i \in I_C(r, u)$.

Proof. Let us first consider the case of (i). By Theorem 1 and Table 2, $\text{RSR}(G, r, \overrightarrow{\rho}.u)$ is of -1 -type. Applying Lemma 7.1 we have that $\mathfrak{B}(G, r, \overrightarrow{\rho}.u)$ is a central quantum linear space. Similarly, $\mathfrak{B}(G, r, \overrightarrow{\rho}.u)$ is a central quantum linear space under the other case.

Conversely, if $\mathfrak{B}(G, r, \overrightarrow{\rho}.u)$ is a central quantum linear space over Weyl Group G of exceptional type, then for any $C \in \mathcal{K}_r(G)$, C has to be $\{s_6\}$ with $G = W(E_7)$ or $\{s_6\}$ with $G = W(E_8)$ or $\{s_2\}$ with $G = W(F_4)$ or $\{s_4\}$ with $G = W(G_2)$ by Lemma 7.1. This implies $r = r_C C$ and C is one case in this theorem. Furthermore, every bi-one type $\text{RSR}(G, \mathcal{O}_{u(C)}, \rho_C^{(i)})$ for any $i \in I_C(r, u)$ is of -1 -type by Proposition 2.6. Applying Theorem 1 and Table 2, Table 5, Table 11 and Table 12, we have that $\chi_C^{(i)}$ has to be one case in this theorem for any $i \in I_C(r, u)$. \square

In other words we have

Remark 7.2. *Let G be a Weyl Group of exceptional type and $M = M(\mathcal{O}_a, \rho^{(1)}) \oplus M(\mathcal{O}_a, \rho^{(2)}) \oplus \cdots \oplus M(\mathcal{O}_a, \rho^{(m)})$ is a YD module over kG . Then $\mathfrak{B}(M)$ is finite dimensional in the following cases:*

- (i) $G = W(E_7)$, $a = s_6$ and the characters of $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(m)}$ are in $\{\chi_6^{(j)} \mid j = 2, 4, 6, 8, 10, 12, 15, 16, 18, 20, 22, 26, 27, 28, 30, 32, 35, 36, 38, 41, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60\}$.
- (ii) $G = W(E_8)$, $a = s_7$ and the characters of $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(m)}$ are in $\{\chi_7^{(j)} \mid j = 3, 4, 11, 12, 16, 17, 18, 19, 29, 30, 32, 33, 34, 37, 38, 45, 46, 51, 52, 56, 57, 60, 63, 64, 65, 66, 71, 79, 80, 82, 83, 89, 90, 91, 92, 95, 96, 99, 100, 103, 104, 106, 107, 108, 112\}$.
- (iii) $G = W(F_4)$, $a = s_2$ and the characters of $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(m)}$ are in $\{\chi_2^{(j)} \mid j = 9, 10, 11, 12, 16, 17, 18, 19, 25\}$.

(iv) $G = W(G_2)$, as_4 and the characters of $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(m)}$ are in $\{\chi_4^{(3)}, \chi_4^{(4)}, \chi_4^{(5)}\}$.

8 Nichols algebras of reducible YD modules

In this section it is proved that except a few cases Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional.

\mathcal{O}_{s_i} and \mathcal{O}_{s_j} are said to be square-commutative if $stst = tsts$ for any $s \in \mathcal{O}_{s_i}, t \in \mathcal{O}_{s_j}$. a and b are said to be square-commutative if $abab = baba$.

Lemma 8.1. *Let G be a Weyl group of Exceptional Type.*

- (i) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are not commutative for any i, j with $i, j \neq 1$ when $G = W(E_6)$.
- (ii) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are not square-commutative when $G = W(E_7)$ and $(i, j) \neq (9, 11), (9, 13), (11, 19), (13, 19)$ with $i, j \neq 1, 6$.
- (iii) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are not square-commutative when $G = W(E_8)$ and $(i, j) \neq (5, 14), (5, 24), (8, 14), (8, 24), (14, 35), (14, 80), (24, 35), (24, 80)$ with $i, j \neq 1, 7$.
- (iv) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are not square-commutative when $G = W(F_4)$ and $(i, j) \neq (3, 3), (3, 4), (3, 7), (3, 8), (3, 17), (3, 18), (3, 24), (3, 25), (4, 4), (4, 7), (4, 8), (4, 17), (4, 18), (4, 24), (4, 25), (7, 12), (7, 13), (7, 17), (7, 18), (8, 12), (8, 13), (8, 17), (8, 18), (12, 17), (12, 18), (13, 17), (13, 18)$ with $i, j \neq 1, 2$.
- (v) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are not square-commutative when $G = W(G_2)$ and $(i, j) \neq (2, 5), (2, 6), (3, 5), (3, 6), (5, 5), (5, 6), (6, 6)$ with $i, j \neq 1, 4$.

Proof. Let $A := \{(i, j) \mid (i, j) = (9, 11), (9, 13), (11, 19), (13, 19), \text{ or } i, j = 1, 6\}$, $B := \{(i, j) \mid (i, j) = (5, 14), (5, 24), (8, 14), (8, 24), (14, 35), (14, 80), (24, 35), (24, 80), \text{ or } i, j = 1, 7\}$, $C := \{(i, j) \mid (i, j) = (3, 3), (3, 4), (3, 7), (3, 8), (3, 17), (3, 18), (3, 24), (3, 25), (4, 4), (4, 7), (4, 8), (4, 17), (4, 18), (4, 24), (4, 25), (7, 12), (7, 13), (7, 17), (7, 18), (8, 12), (8, 13), (8, 17), (8, 18), (12, 17), (12, 18), (13, 17), (13, 18), \text{ or } i, j = 1, 2\}$.

(i) It follows from Table 13.

(ii) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are square-commutative in $W(E_7)$ for $(i, j) \in A$. s_i and s_j are not square-commutative if $(i, j) \notin A$ and there does not exist t such that s_i and $s_t s_j s_t^{-1}$ are in table 14–16.

(iii) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are square-commutative in $W(E_8)$ for $(i, j) \in B$. s_i and $s_{110} s_j s_{110}^{-1}$ in $W(E_8)$ are not square-commutative if $(i, j) \notin B$ and there does not exist t such that s_i and $s_t s_j s_t^{-1}$ are in table 17.

(iv) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are square-commutative in $W(F_4)$ for $(i, j) \in C$. s_i and $s_3 s_j s_3^{-1}$ are not square-commutative in $W(F_4)$ if $(i, j) \notin C$ and there does not exist t such that s_i and $s_t s_j s_t^{-1}$ are in table 18.

(v) \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are square-commutative in $W(G_2)$ for any (i, j) but $(i, j) = (2, 3), (2, 2), (3, 3)$. s_2 and $s_5 s_3 s_5^{-1}$, s_2 and $s_6 s_2 s_6^{-1}$, s_3 and $s_5 s_3 s_5^{-1}$ are not square-commutative, respectively. \square

Note that we have proved that \mathcal{O}_{s_i} and \mathcal{O}_{s_j} are square-commutative in $G = (W(E_7))$, $G = (W(E_8))$ and $G = (W(F_2))$ if and only if $(i, j) \in A, B, C$, respectively. The programs to prove that \mathcal{O}_{s_i} and \mathcal{O}_{s_j} in $W(E_7)$ are square-commutative are the following:

```
gap> L:=SimpleLieAlgebra("E",7,Rationals);;
gap> R:=RootSystem(L);;
gap> W:=WeylGroup(R);;
gap> ccl:=ConjugacyClasses(W);
gap> q:=NrConjugacyClasses(W);;Display (q);
gap> con1:=Elements(ccl[11]);;m:=Size(con1);
gap> for k in [1..m] do
> s:=con1[k];
> con2:=Elements(ccl[19]);n:=Size(con2);
> for l in [1..n] do
> t:=con2[l];
> if (s*t)^2 = (t*s)^2 then
> Print( " k=",k," AND l=",l, " \n");
> fi;
> od;
> od;
```

For any reducible YD module M over kG , there are at least two irreducible YD submodules of M . Therefore we only consider the direct sum of two irreducible YD modules.

We give the final main result.

Theorem 3. *Let G be a Weyl group of Exceptional Type. Then $\dim(\mathfrak{B}(M(\mathcal{O}_{s_i}, \rho^{(1)}) \oplus M(\mathcal{O}_{s_j}, \rho^{(2)})) = \infty$ in the following cases:*

- (i) $G = W(E_6)$.
- (ii) $G = W(E_7)$ and $(i, j) \neq (9, 11), (9, 13), (11, 19), (13, 19)$ and $i, j \neq 6$.
- (iii) $G = W(E_8)$ and $(i, j) \neq (8, 14), (8, 24), (14, 35), (14, 80), (24, 35), (24, 80)$ and $i, j \neq 7$.
- (iv) $G = W(F_4)$ and $(i, j) \neq (3, 3), (3, 4), (3, 7), (3, 8), (3, 17), (3, 18), (3, 24), (3, 25), (4, 4), (4, 7), (4, 8), (4, 17), (4, 18), (4, 24), (4, 25), (7, 13), (7, 17), (7, 18), (8, 13), (8, 17), (8, 18), (13, 17), (13, 18)$ and $i, j \neq 2$.
- (v) $G = W(G_2)$ and $(i, j) \neq (2, 5), (3, 5), (5, 5)$ and $i, j \neq 4$.

Proof. It follows from [HS, Theorem 8.2, Theorem 8.6] and Lemma 8.1. Note that the orders of s_{12} in $W(F_4)$, s_5 in $W(E_8)$ and s_6 in $W(G_2)$ are odd. \square

E_6	
s_i	s_i and $s_t s_j s_t^{-1}$ are not commutative
s_2	$s_7 s_2 s_7^{-1}, s_7 s_3 s_7^{-1}, s_7 s_4 s_7^{-1}, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_5 s_{15} s_5^{-1},$ $s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_3	$s_7 s_3 s_7^{-1}, s_7 s_4 s_7^{-1}, s_7 s_5 s_7^{-1}, s_8 s_6 s_8^{-1}, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}$ $s_{14}, s_7 s_{15} s_7^{-1}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_4	$s_7 s_4 s_7^{-1}, s_7 s_5 s_7^{-1}, s_8 s_6 s_8^{-1}, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}$ $s_{14}, s_7 s_{15} s_7^{-1}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_5	$s_2 s_5 s_2^{-1}, s_2 s_6 s_2^{-1}, s_2 s_7 s_2^{-1}, s_2 s_8 s_2^{-1}, s_2 s_9 s_2^{-1}, s_2 s_{10} s_2^{-1}, s_{11}, s_{12}, s_{13}$ $s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_2 s_{20} s_2^{-1}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_6	$s_2 s_6 s_2^{-1}, s_2 s_7 s_2^{-1}, s_2 s_8 s_2^{-1}, s_2 s_9 s_2^{-1}, s_2 s_{10} s_2^{-1}, s_{11}, s_{12}, s_{12}$ $s_{14}, s_{15}, s_{16}, s_5 s_{17} s_5^{-1}, s_{18}, s_{19}, s_2 s_{20} s_2^{-1}, s_{21}, s_5 s_{22} s_5^{-1}, s_{23}, s_{24}, s_5 s_{25} s_5^{-1},$
s_7	$s_2 s_7 s_2^{-1}, s_2 s_8 s_2^{-1}, s_9, s_{10}, s_{11}, s_{12}, s_{13}$ $s_{14}, s_2 s_{15} s_2^{-1}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_8	$s_2 s_8 s_2^{-1}, s_2 s_9 s_2^{-1}, s_3 s_{10} s_3^{-1}, s_2 s_{11} s_2^{-1}, s_2 s_{12} s_2^{-1}, s_{13}$ $s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_2 s_{20} s_2^{-1}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_9	$s_2 s_9 s_2^{-1}, s_2 s_{10} s_2^{-1}, s_2 s_{11} s_2^{-1}, s_2 s_{12} s_2^{-1}, s_{13}$ $s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_2 s_{20} s_2^{-1}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{10}	$s_2 s_{10} s_2^{-1}, s_2 s_{11} s_2^{-1}, s_2 s_{12} s_2^{-1}, s_2 s_{13} s_2^{-1}, s_2 s_{14} s_2^{-1},$ $s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_2 s_{20} s_2^{-1}, s_{21}, s_{22}, s_{23}, s_{24}, s_2 s_{25} s_2^{-1}$
s_{11}	$s_2 s_{11} s_2^{-1}, s_2 s_{12} s_2^{-1}, s_2 s_{13} s_2^{-1}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{12}	$s_2 s_{12} s_2^{-1}, s_8 s_{13} s_8^{-1}, s_2 s_{14} s_2^{-1}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{13}	$s_2 s_{13} s_2^{-1}, s_2 s_{14} s_2^{-1}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{14}	$s_2 s_{14} s_2^{-1}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{15}	$s_5 s_{15} s_5^{-1}, s_5 s_{16} s_5^{-1}, s_{17}, s_8 s_{18} s_8^{-1}, s_8 s_{19} s_8^{-1}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{16}	$s_2 s_{16} s_2^{-1}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{17}	$s_2 s_{17} s_2^{-1}, s_2 s_{18} s_2^{-1}, s_{19}, s_{20}, s_{21}, s_2 s_{22} s_2^{-1}, s_{23}, s_{24}, s_{25}$
s_{18}	$s_2 s_{18} s_2^{-1}, s_8 s_{19} s_8^{-1}, s_{20}, s_{21}, s_{22} s_{23}, s_{24}, s_{25}$
s_{19}	$s_2 s_{19} s_2^{-1}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{20}	$s_2 s_{20} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_{22}, s_{23}, s_{24}, s_{25}$
s_{21}	$s_2 s_{21} s_2^{-1}, s_8 s_{22} s_8^{-1}, s_2 s_{23} s_2^{-1}, s_2 s_{24} s_2^{-1}, s_{25}$
s_{22}	$s_{11} s_{22} s_{11}^{-1}, s_8 s_{23} s_8^{-1}, s_8 s_{24} s_8^{-1}, s_{25}$
s_{23}	$s_2 s_{23} s_2^{-1}, s_2 s_{24} s_2^{-1}, s_{25}$
s_{24}	$s_2 s_{24} s_2^{-1}, s_{25}$
s_{25}	$s_2 s_{25} s_2^{-1}$

Table 13

E_7	
s_i	s_i and $s_t s_j s_t^{-1}$ are not square-commutative
s_2	$s_{60} s_2 s_{60}^{-1}, s_{60} s_3 s_{60}^{-1}, s_{60} s_4 s_{60}^{-1}, s_{60} s_5 s_{60}^{-1},$
s_3	$s_{60} s_3 s_{60}^{-1}, s_{60} s_4 s_{60}^{-1}, s_{60} s_5 s_{60}^{-1},$
s_4	$s_{60} s_4 s_{60}^{-1}, s_{59} s_5 s_{59}^{-1}, s_{59} s_9 s_{59}^{-1}, s_{44} s_{11} s_{44}^{-1}, s_{44} s_{13} s_{44}^{-1}$
s_5	$s_{60} s_5 s_{60}^{-1}, s_{44} s_9 s_{44}^{-1}, s_{44} s_{11} s_{44}^{-1}, s_{44} s_{13} s_{44}^{-1}, s_{44} s_{34} s_{44}^{-1}, s_{44} s_{57} s_{44}^{-1}$
s_7	$s_{60} s_7 s_{60}^{-1}, s_{44} s_8 s_{44}^{-1}, s_{44} s_9 s_{44}^{-1}, s_{44} s_{10} s_{44}^{-1}, s_{44} s_{11} s_{44}^{-1}, s_{44} s_{12} s_{44}^{-1}, s_{44} s_{13} s_{44}^{-1},$ $s_{44} s_{14} s_{44}^{-1}, s_{44} s_{15} s_{44}^{-1}, s_{44} s_{21} s_{44}^{-1}$
s_8	$s_{60} s_8 s_{60}^{-1}, s_{44} s_9 s_{44}^{-1}, s_{44} s_{10} s_{44}^{-1}, s_{44} s_{11} s_{44}^{-1}, s_{44} s_{12} s_{44}^{-1}, s_{44} s_{13} s_{44}^{-1}, s_{44} s_{14} s_{44}^{-1},$ $s_{44} s_{15} s_{44}^{-1}, s_{44} s_{16} s_{44}^{-1}, s_{44} s_{18} s_{44}^{-1}$
s_9	$s_2 s_9 s_2^{-1}, s_{44} s_{10} s_{44}^{-1}, s_{44} s_{12} s_{44}^{-1}, s_2 s_{14} s_2^{-1}, s_2 s_{15} s_2^{-1}, s_2 s_{16} s_2^{-1},$ $s_2 s_{17} s_2^{-1}, s_2 s_{18} s_2^{-1}, s_2 s_{19} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_2 s_{24} s_2^{-1}, s_2 s_{26} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{10}	$s_2 s_{10} s_2^{-1}, s_2 s_{11} s_2^{-1}, s_2 s_{12} s_2^{-1}, s_2 s_{13} s_2^{-1}, s_2 s_{14} s_2^{-1}, s_2 s_{15} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{11}	$s_3 s_{11} s_3^{-1}, s_2 s_{12} s_2^{-1}, s_3 s_{13} s_3^{-1}, s_2 s_{14} s_2^{-1}, s_2 s_{15} s_2^{-1}, s_2 s_{16} s_2^{-1}, s_3 s_{17} s_3^{-1}, s_3 s_{18} s_3^{-1},$ $s_2 s_{20} s_2^{-1}, s_3 s_{21} s_3^{-1}, s_2 s_{25} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_2 s_{27} s_2^{-1}, s_2 s_{28} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_3 s_{30} s_3^{-1},$ $s_3 s_{31} s_3^{-1}, s_2 s_{34} s_2^{-1}, s_2 s_{36} s_2^{-1}, s_2 s_{37} s_2^{-1}, s_2 s_{38} s_2^{-1}, s_2 s_{39} s_2^{-1}, s_2 s_{40} s_2^{-1}, s_2 s_{41} s_2^{-1},$ $s_2 s_{44} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{54} s_2^{-1}, s_2 s_{56} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{59} s_2^{-1}, s_2 s_{60} s_2^{-1}$
s_{12}	$s_2 s_{12} s_2^{-1}, s_2 s_{13} s_2^{-1}, s_2 s_{14} s_2^{-1}, s_2 s_{15} s_2^{-1}, s_2 s_{16} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{13}	$s_3 s_{13} s_3^{-1}, s_2 s_{14} s_2^{-1}, s_2 s_{15} s_2^{-1}, s_2 s_{16} s_2^{-1}, s_3 s_{17} s_3^{-1}, s_3 s_{18} s_3^{-1}, s_2 s_{20} s_2^{-1}, s_3 s_{21} s_3^{-1},$ $s_2 s_{25} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_2 s_{27} s_2^{-1}, s_2 s_{28} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_3 s_{30} s_3^{-1}, s_3 s_{31} s_3^{-1},$ $s_2 s_{34} s_2^{-1}, s_2 s_{36} s_2^{-1}, s_2 s_{37} s_2^{-1}, s_2 s_{38} s_2^{-1}, s_2 s_{39} s_2^{-1}, s_2 s_{40} s_2^{-1}, s_2 s_{41} s_2^{-1},$ $s_2 s_{44} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{54} s_2^{-1}, s_2 s_{56} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{59} s_2^{-1}, s_2 s_{60} s_2^{-1}$
s_{14}	$s_2 s_{14} s_2^{-1}, s_2 s_{15} s_2^{-1}, s_2 s_{16} s_2^{-1}, s_2 s_{17} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_2 s_{25} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{36} s_2^{-1},$ $s_2 s_{37} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{15}	$s_3 s_{15} s_3^{-1}, s_2 s_{16} s_2^{-1}, s_2 s_{17} s_2^{-1}, s_2 s_{18} s_2^{-1}, s_2 s_{27} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{36} s_2^{-1}, s_2 s_{37} s_2^{-1},$ $s_2 s_{42} s_2^{-1}, s_2 s_{43} s_2^{-1}, s_2 s_{56} s_2^{-1}$
s_{16}	$s_2 s_{16} s_2^{-1}, s_2 s_{17} s_2^{-1}, s_2 s_{18} s_2^{-1}, s_3 s_{19} s_3^{-1}, s_2 s_{20} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_2 s_{28} s_2^{-1},$ $s_2 s_{29} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{55} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{58} s_2^{-1}$
s_{17}	$s_3 s_{17} s_3^{-1}, s_3 s_{18} s_3^{-1}, s_{23} s_{19} s_{23}^{-1}, s_2 s_{20} s_2^{-1}, s_3 s_{21} s_3^{-1}, s_2 s_{25} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_2 s_{28} s_2^{-1},$ $s_2 s_{29} s_2^{-1}, s_3 s_{30} s_3^{-1}, s_3 s_{31} s_3^{-1}, s_2 s_{36} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{56} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{60} s_2^{-1},$
s_{18}	$s_2 s_{18} s_2^{-1}, s_3 s_{19} s_3^{-1}, s_2 s_{20} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_2 s_{28} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{30} s_2^{-1},$ $s_2 s_{44} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{57} s_2^{-1}$
s_{19}	$s_3 s_{19} s_3^{-1}, s_2 s_{20} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_3 s_{27} s_3^{-1}, s_2 s_{28} s_2^{-1},$ $s_2 s_{29} s_2^{-1}, s_2 s_{30} s_2^{-1}, s_2 s_{31} s_2^{-1}, s_2 s_{32} s_2^{-1}, s_2 s_{33} s_2^{-1}, s_2 s_{44} s_2^{-1}, s_2 s_{46} s_2^{-1}, s_2 s_{54} s_2^{-1},$ $s_2 s_{55} s_2^{-1}, s_3 s_{57} s_3^{-1}, s_2 s_{58} s_2^{-1}, s_2 s_{59} s_2^{-1}, s_2 s_{60} s_2^{-1}$

Table 14

E_7	
s_{20}	$s_2 s_{20} s_2^{-1}, s_2 s_{21} s_2^{-1}, s_2 s_{26} s_2^{-1}, s_2 s_{27} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{55} s_2^{-1}, s_2 s_{58} s_2^{-1}$
s_{21}	$s_2 s_{21} s_2^{-1}, s_3 s_{26} s_3^{-1}, s_2 s_{28} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{30} s_2^{-1}, s_2 s_{32} s_2^{-1}, s_2 s_{40} s_2^{-1}, s_2 s_{44} s_2^{-1},$ $s_2 s_{55} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{58} s_2^{-1}$
s_{22}	$s_2 s_{22} s_2^{-1}, s_2 s_{23} s_2^{-1}, s_2 s_{24} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{23}	$s_2 s_{23} s_2^{-1}, s_2 s_{24} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{24}	$s_2 s_{24} s_2^{-1}, s_2 s_{45} s_2^{-1}$
s_{25}	$s_2 s_{25} s_2^{-1}, s_2 s_{49} s_2^{-1}, s_2 s_{56} s_2^{-1}, s_2 s_{57} s_2^{-1}$
s_{26}	$s_3 s_{26} s_3^{-1}, s_2 s_{27} s_2^{-1}, s_2 s_{28} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{30} s_2^{-1}, s_2 s_{31} s_2^{-1}, s_2 s_{44} s_2^{-1},$ $s_2 s_{54} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{59} s_2^{-1}, s_2 s_{60} s_2^{-1}$
s_{27}	$s_2 s_{27} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{38} s_2^{-1}, s_2 s_{39} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{59} s_2^{-1}$
s_{28}	$s_2 s_{28} s_2^{-1}, s_2 s_{29} s_2^{-1}, s_2 s_{30} s_2^{-1}, s_2 s_{57} s_2^{-1}$
s_{29}	$s_2 s_{29} s_2^{-1}, s_2 s_{30} s_2^{-1}, s_2 s_{31} s_2^{-1}, s_2 s_{32} s_2^{-1}, s_2 s_{33} s_2^{-1}, s_2 s_{34} s_2^{-1}, s_2 s_{36} s_2^{-1}, s_2 s_{45} s_2^{-1}, s_2 s_{59} s_2^{-1}$
s_{30}	$s_2 s_{30} s_2^{-1}, s_2 s_{31} s_2^{-1}, s_2 s_{34} s_2^{-1}$
s_{31}	$s_2 s_{31} s_2^{-1}$
s_{32}	$s_2 s_{32} s_2^{-1}, s_2 s_{33} s_2^{-1}, s_2 s_{46} s_2^{-1}, s_2 s_{55} s_2^{-1}, s_2 s_{58} s_2^{-1}$
s_{33}	$s_2 s_{33} s_2^{-1}, s_2 s_{46} s_2^{-1}$
s_{34}	$s_2 s_{34} s_2^{-1}, s_2 s_{35} s_2^{-1}, s_2 s_{53} s_2^{-1}, s_2 s_{59} s_2^{-1}$
s_{35}	$s_2 s_{35} s_2^{-1}, s_2 s_{53} s_2^{-1}$
s_{36}	$s_2 s_{36} s_2^{-1}, s_2 s_{37} s_2^{-1}, s_2 s_{43} s_2^{-1}$
s_{37}	$s_2 s_{37} s_2^{-1}, s_3 s_{43} s_3^{-1}$
s_{38}	$s_2 s_{38} s_2^{-1}, s_2 s_{39} s_2^{-1}$
s_{39}	$s_2 s_{39} s_2^{-1}$
s_{40}	$s_2 s_{40} s_2^{-1}, s_2 s_{41} s_2^{-1}$
s_{41}	$s_2 s_{41} s_2^{-1}, s_2 s_{59} s_2^{-1}$
s_{42}	$s_2 s_{42} s_2^{-1}$
s_{43}	$s_2 s_{43} s_2^{-1}$
s_{44}	$s_2 s_{44} s_2^{-1}$
s_{45}	$s_2 s_{45} s_2^{-1}$
s_{46}	$s_2 s_{46} s_2^{-1}$
s_{47}	$s_2 s_{47} s_2^{-1}, s_2 s_{48} s_2^{-1}, s_2 s_{49} s_2^{-1}, s_2 s_{50} s_2^{-1}$
s_{48}	$s_2 s_{48} s_2^{-1}, s_2 s_{49} s_2^{-1}, s_2 s_{50} s_2^{-1}$
s_{49}	$s_2 s_{49} s_2^{-1}, s_2 s_{50} s_2^{-1}$
s_{50}	$s_2 s_{50} s_2^{-1}$
s_{51}	$s_2 s_{51} s_2^{-1}, s_2 s_{52} s_2^{-1}$
s_{52}	$s_2 s_{52} s_2^{-1}$

Table 15

E_7	
s_{53}	$s_2 s_{53} s_2^{-1}$
s_{54}	$s_2 s_{54} s_2^{-1}$
s_{55}	$s_2 s_{55} s_2^{-1}, s_2 s_{58} s_2^{-1}$
s_{56}	$s_2 s_{56} s_2^{-1}, s_2 s_{57} s_2^{-1}$
s_{57}	$s_2 s_{57} s_2^{-1}$
s_{58}	$s_2 s_{58} s_2^{-1}$
s_{59}	$s_2 s_{59} s_2^{-1}$
s_{60}	$s_2 s_{60} s_2^{-1}$

Table 16

E_8	
s_i	s_i and $s_t s_j s_t^{-1}$ are not square-commutative
s_5	$s_{41} s_5 s_{41}^{-1}, s_{54} s_{15} s_{54}^{-1}, s_{112} s_{18} s_{112}^{-1}, s_9 s_{26} s_9^{-1}, s_2 s_{38} s_2^{-1}, s_2 s_{106} s_2^{-1},$
s_6	$s_2 s_{12} s_2^{-1}, s_2 s_{44} s_2^{-1}$
s_8	$s_{41} s_8 s_{41}^{-1}, s_{112} s_{12} s_{112}^{-1}, s_2 s_{22} s_2^{-1}, s_9 s_{26} s_9^{-1}, s_2 s_{38} s_2^{-1}$
s_{12}	$s_2 s_{24} s_2^{-1}, s_2 s_{26} s_2^{-1}, s_2 s_{50} s_2^{-1}, s_2 s_{51} s_2^{-1}, s_2 s_{62} s_2^{-1}, s_{41} s_{80} s_{41}^{-1}$
s_{14}	$s_2 s_{14} s_2^{-1}, s_{41} s_{21} s_{41}^{-1}, s_9 s_{26} s_9^{-1}, s_2 s_{32} s_2^{-1}, s_2 s_{38} s_2^{-1}, s_2 s_{39} s_2^{-1}, s_2 s_{53} s_2^{-1}, s_9 s_{56} s_9^{-1},$ $s_9 s_{57} s_9^{-1}, s_2 s_{58} s_2^{-1}, s_9 s_{68} s_9^{-1}, s_{70}, s_9 s_{108} s_9^{-1}$
s_{15}	$s_2 s_{26} s_2^{-1}$
s_{18}	$s_2 s_{24} s_2^{-1}, s_2 s_{26} s_2^{-1}, s_2 s_{62} s_2^{-1}, s_2 s_{74} s_2^{-1}, s_{41} s_{80} s_{41}^{-1}$
s_{21}	$s_{41} s_{21} s_{41}^{-1}, s_2 s_{53} s_2^{-1}$
s_{22}	$s_2 s_{56} s_2^{-1}, s_2 s_{57} s_2^{-1}, s_2 s_{70} s_2^{-1}$
s_{24}	$s_{41} s_{38} s_{41}^{-1}, s_2 s_{39} s_2^{-1}, s_{41} s_{42} s_{41}^{-1}, s_2 s_{51} s_2^{-1}, s_{35} s_{53} s_{35}^{-1}, s_2 s_{105} s_2^{-1}, s_9 s_{106} s_9^{-1}$
s_{26}	$s_9 s_{35} s_9^{-1}, s_2 s_{42} s_2^{-1}, s_2 s_{51} s_2^{-1}, s_2 s_{106} s_2^{-1}$
s_{35}	$s_2 s_{48} s_2^{-1}$
s_{45}	$s_2 s_{75} s_2^{-1}, s_2 s_{80} s_2^{-1}, s_2 s_{108} s_2^{-1}$
s_{80}	$s_2 s_{106} s_2^{-1}$
s_{110}	$s_2 s_{110} s_2^{-1}$

Table 17

F_4	
s_i	s_i and $s_t s_j s_t^{-1}$ are not square-commutative
s_3	$s_{23} s_9 s_{23}^{-1}, s_{23} s_{19} s_{23}^{-1}, s_5 s_{23} s_5^{-1},$
s_4	$s_{14} s_{22} s_{14}^{-1}, s_{20} s_{23} s_{20}^{-1}$
s_5	$s_{23} s_{12} s_{23}^{-1}, s_{23} s_{13} s_{23}^{-1}$
s_6	$s_{23} s_{12} s_{23}^{-1}, s_{23} s_{13} s_{23}^{-1}$
s_7	$s_{14} s_7 s_{14}^{-1}, s_{14} s_8 s_{14}^{-1}, s_{14} s_9 s_{14}^{-1}, s_{14} s_{22} s_{14}^{-1}, s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_8	$s_{14} s_8 s_{14}^{-1}, s_{14} s_9 s_{14}^{-1}, s_{14} s_{22} s_{14}^{-1}, s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_9	$s_{14} s_9 s_{14}^{-1}, s_{14} s_{19} s_{14}^{-1}, s_{23} s_{23} s_{23}^{-1}$
s_{10}	$s_{23} s_{12} s_{23}^{-1}, s_{23} s_{13} s_{23}^{-1}$
s_{11}	$s_{23} s_{12} s_{23}^{-1}, s_{23} s_{13} s_{23}^{-1}$
s_{12}	$s_{23} s_{24} s_{23}^{-1}, s_{23} s_{25} s_{23}^{-1}$
s_{13}	$s_{23} s_{24} s_{23}^{-1}, s_{23} s_{25} s_{23}^{-1}$
s_{14}	$s_{14} s_{22} s_{14}^{-1}, s_{23} s_{23} s_{23}^{-1}$
s_{15}	$s_{23} s_{20} s_{23}^{-1}, s_{23} s_{21} s_{23}^{-1}$
s_{16}	$s_{23} s_{20} s_{23}^{-1}, s_{23} s_{21} s_{23}^{-1}$
s_{17}	$s_{14} s_{17} s_{14}^{-1}, s_{14} s_{18} s_{14}^{-1}, s_{14} s_{19} s_{14}^{-1}, s_{14} s_{22} s_{14}^{-1}, s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_{18}	$s_{14} s_{18} s_{14}^{-1}, s_{14} s_{19} s_{14}^{-1}, s_{14} s_{22} s_{14}^{-1}, s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_{19}	$s_{14} s_{19} s_{14}^{-1}, s_{14} s_{23} s_{14}^{-1}$
s_{22}	$s_{14} s_{22} s_{14}^{-1}, s_{14} s_{23} s_{14}^{-1}, s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_{23}	$s_{14} s_{23} s_{14}^{-1}, s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_{24}	$s_{14} s_{24} s_{14}^{-1}, s_{14} s_{25} s_{14}^{-1}$
s_{25}	$s_{14} s_{25} s_{14}^{-1}$

Table 18

Acknowledgement: We would like to thank Prof. N. Andruskiewitsch and Dr. F. Fantino for suggestions and help. The first author and the second author were financially supported by the Australian Research Council. S.C.Zhang thanks the School of Mathematics and Physics, The University of Queensland for hospitality.

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